X-Ray Imaging, Mathematics, and Puzzles

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Abstract: In this module, students will learn about x-ray imaging and how mathematics plays a role in generating the images that we obtain using x-ray devices. First, students will be given lessons on matrix operations including adding, multiplying, and Gaussian Elimination. Students will participate in several games/puzzles such as Kakuro and Nonogram. The strategy used in these games parallel the mathematical techniques used in generating images from x-ray devices. The students will read an article that ties x-ray imaging to mathematics and participate in a discussion.

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Implementation Notes

- **Length of module:**
  - In total, this unit is designed to take approximately 5 days of 90-minute lessons, or 10 days of 45-minute lessons.
  - Each of the lessons is accompanied by an estimate of the length of time it is designed to take in class. If the estimate is longer than you are able to devote in class, feel free to select portions for students to complete outside of class.

- **Relevant courses:** This module is designed to be self-contained, as the first 3 lessons provide foundational knowledge in the linear algebra skills that students will need for the subsequent lessons. The materials are appropriate for any NC Math 4, Pre-Calculus, or Discrete Mathematics for Computer Science courses. This could also serve as an interesting study following the AP exam for students in AP Calculus AB or BC.

- **Mathematical practices/student learning outcomes:** In addition to the standards for mathematical practices, this module addresses a number of standards covered in NC Math 4, Pre-Calculus and Discrete Mathematics for Computer Science.
  - *Mathematical practices*:
    - Make sense of problems and persevere in solving them.
    - Reason abstractly and quantitatively.
    - Construct viable arguments and critique the reasoning of others.
    - Model with mathematics.
    - Use appropriate tools strategically.
    - Attend to precision.
    - Look for and make use of structure.
    - Look for and express regularity in repeated reasoning.
    - Use strategies and procedures flexibly.
    - Reflect on mistakes and misconceptions.
  - ***NC Math 4:*** NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; NC.M4.N.2.2 Execute procedures of addition, subtraction, and scalar multiplication on vectors.
  - ***Precalculus:*** PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; PC.N.2.5 Execute the multiplication algorithm with matrices.
  - ***Discrete Mathematics for Computer Science*:** DCS.N.1.1 Implement procedures of addition, subtraction, multiplication, and scalar multiplication on matrices;

DCS.N.1.2 Implement procedures of addition, subtraction, and scalar multiplication on vectors; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation

- **Assessments:**
  - Feel free to select portions of the guided notes to serve as out-of-class activities.
  - Any problem set contained within guided notes could be given as homework assignments.
  - The teacher could choose to give students a standard test or quiz on the skills that have been learned.

- **Online delivery suggestions:**
  - For asynchronous online delivery, create instructional videos to take students through the guided notes.
  - For synchronous online delivery, display the guided notes on your screen and take students through the activities while you annotate on your screen (or writing on paper and using a document camera).
  - Share all prepared documents through a learning management system so that students would have access to them at home

- **Student Versions:** Please note that the student versions are located at the end of this document in the Appendix.

Lesson 1: Introduction to Matrices and Matrix Operations

Lesson Plan

<table>
<tr>
<td colspan="2">Standard
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices

PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar;

DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices.</td>
<td colspan="2">Topic/Day: Introduction to Matrices, Matrix Addition/Subtraction, and Scalar Multiplication
Content Objective: Elementary Matrix Operations
Vocabulary: matrix; row; column; dimension; square; transpose
Materials Needed: guided notes and practice problems

(~60 minutes)</td>
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<td></td>
<td>Time</td>
<td>Student Does</td>
<td>Teacher Does</td>
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Warm Up Elicit/Engage Build relevance through a problem Try to find out what your students already know Get them interested	~15 min	Students read the opening problem (e.g., Textbook Problem or other context of interest) from a handout and/or projected on a screen. In groups of 2-3, students talk briefly about how they would answer the question from the teacher. (~5 minutes) The teacher brings back students to share out with the class. (~3 minutes) Students are provided guided notes to document new terms (e.g., matrix, dimension, row, column, etc.) They will complete the notes through the discussion conducted by the teacher. (~5 minutes)	Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, "How might we organize this information in a way that allows us to answer questions about the university's inventory?" After students discuss, the teacher solicits students' responses. If students do not suggest a matrix, the teacher will introduce the name and ask students if they are familiar with the term. If not, the teacher will define it through one of the matrices used to organize the information in the problem.
Explore I Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~15 min	Students complete the second matrix from the problem in their groups. (~5 min) Students work in groups of 2-3 to answer teacher's question. (~5 minutes)	Teacher circulates the room to observe/monitor students' work. Teacher then poses question: "How could we use these matrices to determine the total inventory of books at the university?" Once students have some time to answer question, teacher returns to full class discussion to ask how we could define matrix addition.
Explain I Personalize/Differentiate as needed Adjust along teacher/student centered continuum	~5 min	Students offer their ideas on how to define matrix addition (and subtraction). Students engage in class discussion on teachers' questions.	Teacher conducts discussion on matrix addition and subtraction. Teacher poses questions: "Is matrix addition commutative? Is it associative? Is matrix

Provide vocabulary Clarify understandings			subtraction commutative? Is it associative? Why/why not?"
Explore II Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~5 min	Students work in groups of 2-3 to answer teacher's question. (~5 minutes)	Teacher then poses question: "How could we use these matrices to determine the inventory of books at the university if the librarian would like to double the inventory?" Once students have some time to answer question, teacher returns to full class discussion to ask how we could define scalar multiplication.
Explain II Personalize/Differentiate as needed Clarify understandings	~5 min	Students offer their ideas on how to define scalar multiplication.	Teacher conducts discussion on scalar multiplication.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~15 min	Students complete class problem set in groups of 2-3 to apply their new knowledge.	Teacher circulates the room and observes/monitors students' work.
Evaluate Formative Assessment How will you know if students understand throughout the lesson?	N/A	Students will complete guided notes and a problem set for practice. Students turn in their solutions to the last problem in the problem set as an exit ticket (e.g., Stereo Problem)	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking.

Guided Notes – Teacher Version

Matrix Addition, Subtraction and Scalar Multiplication

A university is taking inventory of the books they carry at their two biggest bookstores.

The East Campus bookstore carries the following books:

Hardcover: Textbooks-5280; Fiction-1680; NonFiction-2320; Reference-1890

Paperback: Textbooks-1930; Fiction-2705; NonFiction-1560; Reference-2130

The West Campus bookstore carries the following books:

Hardcover: Textbooks-7230; Fiction-2450; NonFiction-3100; Reference-1380

Paperback: Textbooks-1740; Fiction-2420; NonFiction-1750; Reference-1170

In order to work with this information, we can represent the inventory of each bookstore using an organized array of numbers known as a *matrix*.

Definitions: A **matrix** is a rectangular table of entries and is used to organize data in a way that can be used to solve problems. The following is a list of terms used to describe matrices:

- A matrix's **size (or dimension)** is written by listing the number of rows "by" the number of columns.
- The values in a matrix, A, are referred to as **entries** or **elements**. The entry in the "m^{th}" row and "n^{th}" column is written as a_{mn}.
- A matrix is **square** if it has the same number of rows as it has columns.
- If a matrix has only one row, then it is a row **vector**. If it has only one column, then the matrix is a column **vector.**
- The **transpose** of a matrix, A, written A^T, switches the rows with the columns of A and the columns with the rows.
- Two matrices are **equal** if they have the same size and the same corresponding entries.

The inventory of the books at the East Campus bookstore can be represented with the following **2 x 4** matrix:

$$E = \begin{array}{c c} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} \end{array}$$

Similarly, the West Campus bookstore's inventory can be represented with the following matrix:

$$W = \begin{array}{c c} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} 7230 & 2450 & 3100 & 1380 \\ 1740 & 2420 & 1750 & 1170 \end{bmatrix} \end{array}$$

Adding and Subtracting Matrices

In order to add or subtract matrices, they must first be of the same **size**. The result of the addition or subtraction is a matrix of the same size as the matrices themselves, and the entries are obtained by adding or subtracting the elements in corresponding positions.

In our campus bookstores example, we can find the total inventory between the two bookstores as follows:

$$E + W = \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} + \begin{bmatrix} 7230 & 2450 & 3100 & 1380 \\ 1740 & 2420 & 1750 & 1170 \end{bmatrix}$$

$$= \begin{array}{c c} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} 12510 & 4130 & 5420 & 3270 \\ 3670 & 5125 & 3310 & 3300 \end{bmatrix} \end{array}$$

Question: *Is matrix addition commutative* $(e.g., A + B = B + A)$? *Why or why not?*
Matrix addition is commutative. This is because the operation is based in the addition of real numbers, as the entries of each matrix are added to their corresponding entries in the other matrix/matrices. Since addition of real numbers is commutative, so is matrix addition.

Question: *Is matrix subtraction commutative* $(e.g., A - B = B - A)$? *Why or why not?*
Matrix subtraction is not commutative. This is because the operation is based in the subtraction of real numbers, as the entries of each matrix are subtracted from their corresponding entries in the other matrix/matrices. Since subtraction of real numbers is not commutative, neither is matrix subtraction.

Question: *Is matrix addition associative* $(e.g., (A + B) + C = A + (B + C))$? *Why or why not?*
Matrix addition is associative. This is because the operation is based in the addition of real numbers, as the entries of each matrix are added to their corresponding entries in the other matrix/matrices. Since addition of real numbers is associative, so is matrix addition.

Question: *Is matrix subtraction associative* $(e.g., (A - B) - C = A - (B - C))$? *Why or why not?*
Matrix subtraction is not associative. This is because the operation is based in the subtraction of real numbers, as the entries of each matrix are subtracted from their corresponding entries in the other matrix/matrices. Since subtraction of real numbers is not associative, neither is matrix subtraction.

Scalar Multiplication

Multiplying a matrix by a constant (or *scalar*) is as simple as multiplying each entry by that number! Suppose the bookstore manager in East Campus wants to double his inventory. He can find the number of books of each type that he would need by simply multiplying the matrix *E* by the scalar (or constant) 2. The result is as follows:

$$2E = 2 * \begin{array}{cccc} T & F & N & R \\ \multicolumn{4}{c}{\begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix}} \end{array} = \begin{array}{cccc} T & F & N & R \\ \multicolumn{4}{c}{\begin{bmatrix} 2(5280) & 2(1680) & 2(2320) & 2(1890) \\ 2(1930) & 2(2705) & 2(1560) & 2(2130) \end{bmatrix}} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{r} Hardback \\ Paperback \end{array} & \begin{bmatrix} 10560 & 3360 & 4640 & 3780 \\ 3860 & 5410 & 3120 & 4260 \end{bmatrix} \end{array}$$

Exercises: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed:

a. $A + B$: This operation cannot be performed, since matrices A and B are of different dimensions.

b. $B - A$: This operation also cannot be performed, as A and B have different dimensions.

c. $A - C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-6} & \mathbf{22} \\ \mathbf{0} & \mathbf{-8} & \mathbf{12} \\ \mathbf{-11} & \mathbf{8} & \mathbf{7} \end{bmatrix}$

d. $C - A = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{6} & \mathbf{-22} \\ \mathbf{0} & \mathbf{8} & \mathbf{-12} \\ \mathbf{11} & \mathbf{-8} & \mathbf{-7} \end{bmatrix}$

e. $5B = 5 * \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{40} & \mathbf{-30} \end{bmatrix}$

f. $-A + 4C = -\begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} + 4 * \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} =$

$$\begin{bmatrix} -1 & 0 & -1 \\ -2 & 4 & -3 \\ 6 & -1 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 24 & -84 \\ 8 & 16 & -36 \\ 20 & -28 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{24} & \mathbf{-85} \\ \mathbf{6} & \mathbf{20} & \mathbf{-39} \\ \mathbf{26} & \mathbf{-29} & \mathbf{-4} \end{bmatrix}$$

g. $B - D$: This operation cannot be performed, since B and D are not of the same size.

h. $2C - 6A = 2 * \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} - 6 * \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} =$

$$\begin{bmatrix} 0 & 12 & -42 \\ 4 & 8 & -18 \\ 10 & -14 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 6 \\ 12 & -24 & 18 \\ -36 & 6 & 48 \end{bmatrix} = \begin{bmatrix} \mathbf{-6} & \mathbf{12} & \mathbf{-48} \\ \mathbf{-8} & \mathbf{32} & \mathbf{-36} \\ \mathbf{46} & \mathbf{-20} & \mathbf{-46} \end{bmatrix}$$

i. $B^T + D = \begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{7} \\ \mathbf{6} \\ \mathbf{-3} \end{bmatrix}$

Lesson 2: Matrix Multiplication

Lesson Plan

Standard	**Topic/Day: Matrix Multiplication**
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context	**Content Objective: Elementary Matrix Operations** **(~70 minutes)**

	Time	**Student Does**	**Teacher Does**
Warm Up Elicit/Engage Build relevance through a problem Try to find out what your students already know Get them interested	~5 min	Students read the opening problem (e.g., Opera Problem or other context of interest) from a handout and/or projected on a screen.	Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, "How might we organize this information in a way that allows us to answer the question?"

Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~15 min	In groups of 2-3, students work together to calculate each value of interest by hand (not using any specific method). (~10 minutes) Students work in groups of 2-3 to answer teacher's question. (~10 minutes). Students can break the work up among their group members.	Teacher asks students to calculate each value of interest by hand, showing their work but not using any specific method. The teacher brings students back to share their results and confirm their results with other groups.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~15 min	Students follow along the teachers' explanation on their opening problem. Students share their thoughts on teacher's posed questions.	Teacher conducts lesson on matrix multiplication using the opening problem to demonstrate the operation. Teacher poses questions: "Is matrix multiplication commutative? Is it associative? Why/why not?" Teacher provides examples of why they are/aren't, and students practice the operation with those examples.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~25 min	Students complete class problem set in groups of 2-3 to apply their new knowledge.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate Formative Assessment How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.

Guided Notes Lesson 2 – Teacher Version

Matrix Multiplication

The Metropolitan Opera is planning its last cross-country tour. It plans to perform *Carmen* and *La Traviata* in Atlanta in May. The person in charge of logistics wants to make plane reservations for the two troupes. *Carmen* has 2 stars, 25 other adults, 5 children, and 5 staff members. *La Traviata* has 3 stars, 15 other adults, and 4 staff members. There are 3 airlines to choose from. Redwing charges round-trip fares to Atlanta of $630 for first class, $420 for coach, and $250 for youth. Southeastern charges $650 for first class, $350 for coach, and $275 for youth. Air Atlanta charges $700 for first class, $370 for coach, and $150 for youth. Assume stars travel first class, other adults and staff travel coach, and children travel for the youth fare.

Use multiplication and addition to find the total cost for each troupe to travel each of the airlines.

Carmen/Redwing: $2(630) + 30(420) + 5(250) = \$15110$

Carmen/Southeastern: $2(650) + 30(350) + 5(275) = \$13175$

Carmen/Air Atlanta: $2(700) + 30(370) + 5(150) = \$13250$

La Traviata/Redwing: $3(630) + 19(420) + 0(250) = \$9870$

La Traviata/Southeastern: $3(650) + 19(350) + 0(275) = \$8600$

La Traviata/Air Atlanta: $3(700) + 19(370) + 0(150) = \$9130$

It turns out that we can solve problems like these using a matrix operation, specifically **matrix multiplication**!

We first note that matrix multiplication is only defined for matrices of certain sizes. For the product AB of matrices A and B, where A is an $m\ x\ n$ matrix, B must have the same number of rows as A has columns. So, B must have size $\boldsymbol{n}\ x\ p$. The product AB will have size $\boldsymbol{m\ x\ p}$**.**

Exercises

The following is a set of abstract matrices (without row and column labels):

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad O = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 5 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

List at least 5 orders of pairs of matrices from this set for which the product is defined. State the dimension of each product.

MO: 2x1 **MP: 2x2** **PM: 2x2** **MR: 2x2** **RM: 2x2** **NQ: 3x1**

NU: 3x4 **PO: 2x1** **US: 3x2** **UT: 3x1**

Back to the opera…

Define two matrices that organize the information given:

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{stars} & \textbf{adults} & \textbf{children} \end{array} \\ \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} & \begin{bmatrix} 2 & 30 & 5 \\ 3 & 19 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} & \begin{bmatrix} 630 & 650 & 700 \\ 420 & 350 & 370 \\ 250 & 275 & 150 \end{bmatrix} \end{array}$$

We can multiply these two matrices to obtain the same answers we obtained above, all in one matrix!

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{stars} & \textbf{adults} & \textbf{children} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 2 & 30 & 5 \\ 3 & 19 & 0 \end{bmatrix} \end{array} \cdot \begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} & \begin{bmatrix} 630 & 650 & 700 \\ 420 & 350 & 370 \\ 250 & 275 & 150 \end{bmatrix} \end{array}$$

$$= \begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 15110 & 13175 & 13250 \\ 9870 & 8600 & 9130 \end{bmatrix} \end{array}$$

Carmen/Redwing: \$15110

Carmen/Southeastern: \$13175

Carmen/Air Atlanta: \$13250

La Traviata/Redwing: \$9870

La Traviata/Southeastern: \$8600

La Traviata/Air Atlanta: \$9130

Exercises

1. The K.L. Mutton Company has investments in three states - North Carolina, North Dakota, and New Mexico. Its deposits in each state are divided among bonds, mortgages, and consumer loans. The amount of money (in millions of dollars) invested in each category on June 1 is displayed in the table below.

	NC	ND	NM
Bonds	13	25	22
Mort.	6	9	4
Loans	29	17	13

The current yields on these investments are 7.5% for bonds, 11.25% for mortgages, and 6% for consumer loans. Use matrix multiplication to find the total earnings for each state.

Total earnings for each state (in millions of dollars):

$$\begin{array}{ccc} \boldsymbol{Bonds} & \boldsymbol{Mort.} & \boldsymbol{Loans} \end{array} \atop \begin{bmatrix} \mathbf{1.075} & \mathbf{1.1125} & \mathbf{1.06} \end{bmatrix} \quad \begin{array}{c|ccc} & \boldsymbol{NC} & \boldsymbol{ND} & \boldsymbol{NM} \\ \boldsymbol{Bonds} & \mathbf{13} & \mathbf{25} & \mathbf{22} \\ \boldsymbol{Mort.} & \mathbf{6} & \mathbf{9} & \mathbf{4} \\ \boldsymbol{Loans} & \mathbf{29} & \mathbf{17} & \mathbf{13} \end{array} = \begin{array}{ccc} \boldsymbol{NC} & \boldsymbol{ND} & \boldsymbol{NM} \end{array} \atop \begin{bmatrix} \mathbf{3.39} & \mathbf{3.9075} & \mathbf{2.88} \end{bmatrix}$$

2. Several years ago, Ms. Allen invested in growth stocks, which she hoped would increase in value over time. She bought 100 shares of stock A, 200 shares of stock B, and 150 shares of stock C. At the end of each year she records the value of each stock. The table below shows the price per share (in dollars) of stocks A, B, and C at the end of the years 1984, 1985, and 1986.

	1984	1985	1986
Stock A	68.00	72.00	75.00
Stock B	55.00	60.00	67.50
Stock C	82.50	84.00	87.00

Calculate the total value of Ms. Allen's stocks at the end of each year.

Total value of the stocks (in dollars) at the end of each year:

$$\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} \end{array} \atop \begin{bmatrix} \mathbf{100} & \mathbf{200} & \mathbf{150} \end{bmatrix} \quad \begin{array}{c|ccc} & \mathbf{1984} & \mathbf{1985} & \mathbf{1986} \\ \boldsymbol{A} & \mathbf{68} & \mathbf{72} & \mathbf{75} \\ \boldsymbol{B} & \mathbf{55} & \mathbf{60} & \mathbf{67.5} \\ \boldsymbol{C} & \mathbf{82.5} & \mathbf{84} & \mathbf{87} \end{array} = \begin{array}{ccc} \mathbf{1984} & \mathbf{1985} & \mathbf{1986} \end{array} \atop \begin{bmatrix} \mathbf{30,175} & \mathbf{31,800} & \mathbf{34,050} \end{bmatrix}$$

3. The Sound Company produces stereos. Their inventory includes four models - the Budget, the Economy, the Executive, and the President models. The Budget needs 50 transistors, 30 capacitors, 7 connectors, and 3 dials. The Economy model needs 65 transistors, 50 capacitors, 9 connectors, and 4 dials. The Executive model needs 85 transistors, 42 capacitors, 10 connectors, and 6 dials. The President model needs 85 transistors, 42 capacitors, 10 connectors, and 12 dials. The daily manufacturing goal in a normal quarter is 10 Budget, 12 Economy, 11 Executive, and 7 President stereos.

   a. How many transistors are needed each day? Capacitors? Connectors? Dials?

   b. During August and September, production is increased by 40%. How many Budget, Economy, Executive, and President models are produced daily during these months?

   c. It takes 5 person-hours to produce the Budget model, 7 person-hours to produce the Economy model, 6 person-hours for the Executive model, and 7 person-hours for the President model. Determine the number of employees needed to maintain the normal production schedule, assuming everyone works an average of 7 hours each day. How many employees are needed in August and September?

Define the matrices for the inventory parts $(\boldsymbol{I})$ and the daily manufacturing goal $(\boldsymbol{N})$ as

$$\boldsymbol{I} = \begin{array}{c} \\ \boldsymbol{B} \\ \boldsymbol{Ec} \\ \boldsymbol{Ex} \\ \boldsymbol{P} \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{t} & \boldsymbol{ca} & \boldsymbol{co} & \boldsymbol{d} \end{array} \\ \begin{bmatrix} \mathbf{50} & \mathbf{30} & \mathbf{7} & \mathbf{3} \\ \mathbf{65} & \mathbf{50} & \mathbf{9} & \mathbf{4} \\ \mathbf{85} & \mathbf{42} & \mathbf{10} & \mathbf{6} \\ \mathbf{85} & \mathbf{42} & \mathbf{10} & \mathbf{12} \end{bmatrix} \end{array} \quad \boldsymbol{and} \quad \boldsymbol{N} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{B} & \boldsymbol{Ec} & \boldsymbol{Ex} & \boldsymbol{P} \end{array} \\ \begin{bmatrix} \mathbf{10} & \mathbf{12} & \mathbf{11} & \mathbf{7} \end{bmatrix} \end{array}$$

a. The answers are the results of the matrix multiplication

$$\boldsymbol{NI} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{t} & \boldsymbol{ca} & \boldsymbol{co} & \boldsymbol{d} \end{array} \\ \begin{bmatrix} \mathbf{2810} & \mathbf{1656} & \mathbf{358} & \mathbf{228} \end{bmatrix} \end{array}$$

b. The new daily manufacturing goals are given by

$$\mathbf{1.4}\boldsymbol{N} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{B} & \boldsymbol{Ec} & \boldsymbol{Ex} & \boldsymbol{P} \end{array} \\ \begin{bmatrix} \mathbf{14} & \mathbf{16.8} & \mathbf{15.4} & \mathbf{9.8} \end{bmatrix} \end{array}$$

Which should be rounded to integer quantities

c. Define a matrix $\boldsymbol{H}$ for hours of labor as

$$\boldsymbol{H} = \begin{array}{c} \\ \boldsymbol{B} \\ \boldsymbol{Ec} \\ \boldsymbol{Ex} \\ \boldsymbol{P} \end{array} \begin{array}{c} \boldsymbol{Hrs.} \\ \begin{bmatrix} \mathbf{5} \\ \mathbf{7} \\ \mathbf{6} \\ \mathbf{7} \end{bmatrix} \end{array}$$

The number of labor hours needed per week is given by

$$NH = 249$$

With 7-hour workdays, the number of employees needed is $\frac{249}{7} = 35.6$, which implies that 36 employees are needed to maintain full production. For August and September, we want $\frac{1.4NH}{7} = \frac{348.6}{7}$, which rounds to 50.

4. The president of the Lucrative Bank is hoping for a 21% increase in checking accounts, a 35% increase in savings accounts, and a 52% increase in market accounts. The current statistics on the number of accounts at each branch are as follows:

$$\begin{array}{r c} & \begin{array}{ccc} \textit{Checking} & \textit{Savings} & \textit{Market} \end{array} \\ \begin{array}{r} \textit{Northgate} \\ \textit{Downtown} \\ \textit{South Square} \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array}$$

What is the goal for each branch in each type of account? (HINT: multiply by a 3×2 matrix with certain nonzero entries on the diagonal and zero entries elsewhere.) What will be the total number of accounts at each branch?

The goal for each branch in each type of account is given by:

$$\begin{array}{r c} & \begin{array}{ccc} c & s & m \end{array} \\ \begin{array}{r} N \\ S \\ D \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array} \cdot \begin{array}{r c} & \begin{array}{ccc} c & s & m \end{array} \\ \begin{array}{r} c \\ s \\ m \end{array} & \begin{bmatrix} 1.21 & 0 & 0 \\ 0 & 1.35 & 0 \\ 0 & 0 & 1.52 \end{bmatrix} \end{array}$$

$$= \begin{array}{r c} & \begin{array}{ccc} c & s & m \end{array} \\ \begin{array}{r} N \\ S \\ D \end{array} & \begin{bmatrix} 48447 & 13682 & 778.24 \\ 18430 & 11814 & 159.6 \\ 30991 & 16452 & 147.44 \end{bmatrix} \end{array}$$

Right-multiplying this result by the matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields the following total number of accounts at each branch: $\begin{array}{r c} & \textit{Total} \\ \begin{array}{r} N \\ D \\ S \end{array} & \begin{bmatrix} 62907.68 \\ 30402.96 \\ 47590.41 \end{bmatrix} \end{array}$.

Note: this answer can also be obtained by just adding up the entries in each row of the previous matrix.

Lesson 3: Solving Matrix Equations Using Gaussian Elimination

Lesson Plan

<table>
<tr><td colspan="2">Standard
DCS.N.2.1 Organize data into matrices to solve problems;
DCS.N.2.2 Interpret solutions found using matrix operations in context;
DCS.N.2.3 Represent a system of equations as a matrix equation</td><td colspan="2">Topic/Day: Solving Matrix Equations Using Gaussian Elimination
Vocabulary: Gaussian elimination

(~75 minutes)</td></tr>
<tr><td></td><td>Time</td><td>Student Does</td><td>Teacher Does</td></tr>
<tr><td>Warm Up
Elicit/Engage
Build relevance through a problem
Try to find out what your students already know
Get them interested</td><td>~5 min</td><td>Students read the opening problem (e.g., Business Problem or other context of interest) from a handout and/or projected on a screen.

Students work in groups of 2-3 to represent the problem with a system of equations.</td><td>Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, “How might we represent this problem with a system of equations?</td></tr>
<tr><td>Explore
Connectivity to build understanding of concepts
Allow for collaboration consider heterogeneous groups
Move deliberately from concrete to abstract
Apply scaffolding & personalization</td><td>~10 min</td><td>Students work in groups of 2-3 to answer teacher’s question.</td><td>Teacher asks students to consider how they could use matrices to represent the system of equations as a matrix equation.

The teacher brings students back to share their results.</td></tr>
<tr><td>Explain
Personalize/Differentiate as needed
Adjust along teacher/student centered continuum
Provide vocabulary</td><td>~30 min</td><td>Students follow along the teachers’ explanation on a problem out of context. Students work together on practice problems based on the teacher’s lesson.</td><td>Teacher conducts lesson on solving a matrix equation using a non-contextual problem. Teacher introduces the method of Gaussian</td></tr>
</table>

Clarify understandings			Elimination during this part of the lesson. Teacher includes a tutorial on using the calculator to apply Gaussian Elimination.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~20 min	Students apply their new understanding to the opening problem.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.

Guided Notes - Teacher Version

Solving Linear Systems of Equations Using Gaussian Elimination

A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling.

How can we represent this problem with a system of equations?

Let x = amount of money for the East location
Let y = amount of money for the West location
Let z = amount of money for the South location

We therefore have the following system of equations:

$$0.5x + 0.3y + 0.4z = 43{,}000$$
$$0.2x + 0.3y + 0.4z = 28{,}000$$
$$0.3x + 0.4y + 0.2z = 29{,}000$$

Example: Consider the following system of linear equations (recall this from Algebra II):

$$x + 3y = 0$$
$$x + y + z = 1$$
$$3x - y - z = 11$$

We can solve this system by representing it using matrices.

We will name the **coefficient** matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$, the **variable vector** $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the **column vector** $B = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$. So, our **matrix equation** (also referred to as a linear system of equations) representing the system can be written as $AX = B$:

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

One way to solve this system is to use an approach known as **Gaussian elimination**, or **row reduction.**

Gaussian Elimination

You may recall from your prior mathematics work that there are three possible conclusions we can make about the solution to a system of equations.

Case 1: There exists one unique solution.
Case 2: There is no solution.
Case 3: There is an infinite number of solutions.

<u>Case 1</u>: There exists one unique solution.

Recall our example from above:

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

To begin, we write the associated **augmented matrix**, which is written in the following form:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$$

To apply the method on a matrix, we use **elementary row operations** to modify the matrix. Our goal is to end up with the **identity matrix**, which is an $n \; x \; n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, on the left side of the augmented matrix.

Our solution to the system of equations will be the resulting matrix on the right side of the augmented matrix. This is because the resulting augmented matrix would represent a system of equations in which each variable could be solved for (if a solution exists).

Elementary Row Operations:

There are three operations that can be applied to modify the matrix and still preserve the solution to the system of equations.

- Exchanging two rows (which represents the switching the listing order of two equations in the system)
- Multiplying a row by a nonzero scalar (which represents multiplying both sides of one of the equations by a nonzero scalar)
- Adding a multiple of one row to another (which represents does not affect the solution, since both equations are in the system)

For our example…

$$\begin{aligned} x + 3y &= 0 \qquad & R_1 \\ x + y + z &= 1 \qquad & R_2 \\ 3x - y - z &= 11 \qquad & R_3 \end{aligned}$$

System of equations	Row operation	Augmented matrix
$x + 3y = 0$ $x + y + z = 1$ $3x - y - z = 11$		$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $-y + z = 1$ $3x - y - z = 11$	$R_2 - R_1 \to R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $-y + z = 1$ $-10y - z = 11$	$R_3 - 3R_1 \to R_3$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $-12y = 12$ $-10y - z = 11$	$R_2 + R_3 \to R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -12 & 0 & 12 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x + 3y = 0$ $y = -1$ $-10y - z = 11$	$-\frac{1}{12}R_2 \to R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -10 & -1 & 11 \end{array}\right]$

$x = 3$ $y = -1$ $-10y - z = 11$	$R_1 - 3R_2 \rightarrow R_1$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x = 3$ $y = -1$ $-z = 1$	$R_3 + 10R_2 \rightarrow R_3$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$x = 3$ $y = -1$ $z = -1$	$-R_3 \rightarrow R_3$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

The solution to our system is therefore $\boldsymbol{x = 3, y = -1}$ **and** $\boldsymbol{z = -1}$**.**

Back to our opening problem! A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling. How much money will each location receive in grants?

Rewrite your system of equations from earlier in this lesson:

$$0.5x + 0.3y + 0.4z = 43{,}000$$
$$0.2x + 0.3y + 0.4z = 28{,}000$$
$$0.3x + 0.4y + 0.2z = 29{,}000$$

We can represent this system using the following systems of linear equations:

$$\begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 43000 \\ 28000 \\ 29000 \end{bmatrix}$$

The augmented matrix for this system is:

$$\left[\begin{array}{ccc|c} 0.5 & 0.3 & 0.4 & 43000 \\ 0.2 & 0.3 & 0.4 & 28000 \\ 0.3 & 0.4 & 0.2 & 29000 \end{array}\right]$$

Using elementary row operations, we find that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} 50{,}000 \\ 20{,}000 \\ 30{,}000 \end{bmatrix}$$

So, **$50,000** goes to the East location, **$20,000** goes to the West location, and **$30,000** goes to the South location.

Case 2: There is no solution.

Consider the system of equations:

$$\begin{aligned} 2x - y + z &= 1 \\ 3x + 2y - 4z &= 4 \\ -6x + 3y - 3z &= 2 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{-6} & \mathbf{3} & \mathbf{-3} & \mathbf{2} \end{array}\right]$

Using row operation $R_3 + 3R_1 \rightarrow R_3$, we get $\left[\begin{array}{ccc|c} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5} \end{array}\right]$.

We note that the third row in the augmented matrix is a false statement, so there is no solution to this system.

Case 3: There is an infinite number of solutions.

Consider the system of equations:

$$\begin{aligned} x - y + 2z &= -3 \\ 4x + 4y - 2z &= 1 \\ -2x + 2y - 4z &= 6 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{4} & \mathbf{4} & \mathbf{-2} & \mathbf{1} \\ \mathbf{-2} & \mathbf{2} & \mathbf{-4} & \mathbf{6} \end{array}\right]$

Using row operations $R_2 - 4R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$, we get $\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{8} & \mathbf{-10} & \mathbf{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$.

This represents a system that leaves us with 2 equations and 3 unknowns. So, we are unable to solve for one variable without expressing it in terms of another. This gives us an infinite number of solutions.

Exercises

For each of the following problems, identify your variables and write a system of equations to represent the problem. Then use matrices to solve the system.

1. The Frodo Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat is $42 and $30 per acre, respectively. Mr. Frodo has $18,600 available for cultivating these crops. If he wants to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant? (Adapted from *Finite Mathematics,* Tan p. 93 #51[1])

Let x = number of acres of corn
y = number of acres of wheat

$$\mathbf{42x + 30y = 18600}$$
$$\mathbf{x + y = 500}$$

Augmented matrix: $\left[\begin{array}{cc|c} \mathbf{42} & \mathbf{30} & \mathbf{18600} \\ \mathbf{1} & \mathbf{1} & \mathbf{500} \end{array}\right]$

Solution: $x = 300$, $y = 200$

300 acres of corn and 200 acres of wheat should be cultivated.

2. The Coffee Cart sells a blend made with two different coffees, one costing $2.50 per pound, and the other costing $3.00 per pound. If the blended coffee sells for $2.80 per pound, how much of each coffee is used to obtain the blend? (Assume that the weight of the coffee blend is 100 pounds.) (Adapted from *Finite Mathematics,* Tan p. 93 #53)

Let x = number of pounds of $2.50 coffee
y = number of pounds of $3.00 coffee

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[1] Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

$$\mathbf{2.50x + 3.00y = 280}$$
$$\mathbf{x + y = 100}$$

Augmented matrix: $\left[\begin{array}{cc|c} \mathbf{2.5} & \mathbf{3} & \mathbf{280} \\ \mathbf{1} & \mathbf{1} & \mathbf{100} \end{array}\right]$

Solution: $\boldsymbol{x = 40}$, $\boldsymbol{y = 60}$

40 lbs of Coffee 1 should be blended with 60 lbs of Coffee 2 to make the proper blend.

3. The Maple Movie Theater has a seating capacity of 900 and charges \$2 for children, \$3 for students, and \$4 for adults. At a screening with full attendance last week, there were half as many adults as children and students combined. The receipts totaled \$2800. How many adults attended the show? (Adapted from *Finite Mathematics,* Tan p. 97 #60)

Let $\boldsymbol{x}$ **= number of children who attended the show**
$\boldsymbol{y}$ **= number of students who attended the show**
$\boldsymbol{z}$ **= number of adults who attended the show**

$$\mathbf{x + y + z = 2800}$$
$$\mathbf{2x + 3y + 4z = 900}$$
$$\mathbf{x + y - 2z = 0}$$

Augmented matrix: $\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{2800} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{900} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{0} \end{array}\right]$

Solution: $\boldsymbol{x = 200}$, $\boldsymbol{y = 400}$, $\boldsymbol{z = 300}$

200 children, 400 students, and 300 adults attended.

4. The Toolies have a total of \$100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12% per year, while bonds pay 8% per year, and the money market account pays 4% per year. They have decided that the amount invested in stocks should be equal to the difference between the amount invested in bonds and 3 times the amount invested in the money market account. How should the Toolies allocate their resources if they require an annual income of \$10,000 from their investments? (Adapted from *Finite Mathematics,* Tan p. 106 #36)

Let $\boldsymbol{x}$ **= amount allocated to stocks**
$\boldsymbol{y}$ **= amount allocated to bonds**
$\boldsymbol{z}$ **= amount allocated to a money market account**

$$\mathbf{x + y + z = 100{,}000}$$
$$\mathbf{.12x + .08y + .04z = 10{,}000}$$
$$\mathbf{x - y + 3z = 0}$$

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{.12} & \mathbf{.08} & \mathbf{.04} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{100,000} \\ \mathbf{10,000} \\ \mathbf{0} \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{100,000} \\ \mathbf{.12} & \mathbf{.08} & \mathbf{.04} & \mathbf{10,000} \\ \mathbf{1} & \mathbf{-1} & \mathbf{3} & \mathbf{0} \end{array}\right]$

Solution: $\boldsymbol{x = 50,000}$, $\boldsymbol{y = 50,000}$, $\boldsymbol{z = 0}$

$50,000 should be put into the stock market, $50,000 in bonds, and no investment should be made in a Money Market Account.

Lesson 4: Exploring X-Ray Imaging through Puzzles

Lesson Plan

Teacher: Subject: Math

Standard	**Topic/Day:** Puzzles and X-Ray Imaging
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices. DCS.N.2.2 Interpret solutions found using matrix operations in context	**Content Objective:** Use logic to complete Nonograms and Kakuro games that relate to X-Ray imaging **Materials Needed:** Games Printed, Paper, Article: https://plus.maths.org/content/saving-lives-mathematics-tomography **~85 minutes**

	Time	Student Does	Teacher Does
Warm Up Elicit/Engage Build relevance through a problem Try to find out what your students already know Get them interested	~30 mins	Student completes the given puzzles	Actively monitor the students and their progress. Help students when needed but encourage them to utilize each other to work through any difficulties that they may run into.
Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups	~30 mins	Complete a KWL chart about x-rays. Read the article and do a Think, Pair Share with the article.	Observe and ask/answer questions as needed.

Move deliberately from concrete to abstract Apply scaffolding & personalization			
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~25 mins	Participates in discussion around a directed interactive lecture. End the discussion by playing "Guess my Square" (located in the same document). Discuss how there are multiple answers that can be considered correct for the game.	Gives guided notes around the article relating to the mathematics. The optional Milk Delivery Demonstration could go here. When the students are playing "Guess my Square" they should break into teams. You can have them compete locally at their already assigned groups, or you can choose to break the class up into larger teams.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously		Students should complete the challenge puzzles	If you have a highflyer that is easily completed the given puzzles, have some of the challenge puzzles on hand for them to work on.

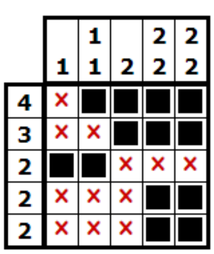
Games KEY

Nonogram and Kakuro Puzzles Name: ____________________________

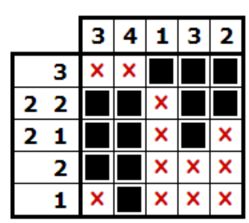
Answer Key Date: _______________ Period: ________

Solve the following Nonogram puzzles[1]. Nonograms are picture logic puzzles in which cells in a grid must be colored or left blank according to the numbers at the side of the grid to reveal a hidden picture. For example: 1 5 2 means 1 square, 5 squares, and 2 squares, in this order, separated by one or more spaces between them.

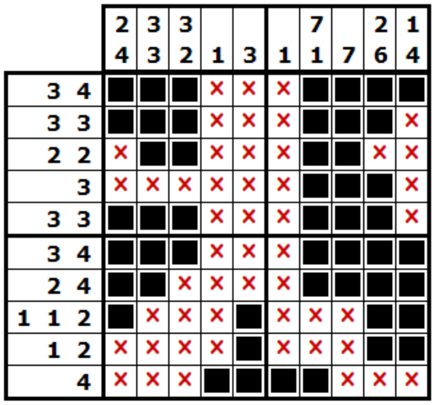
Example:



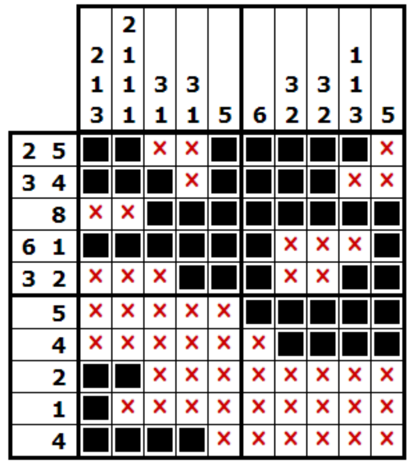
1)



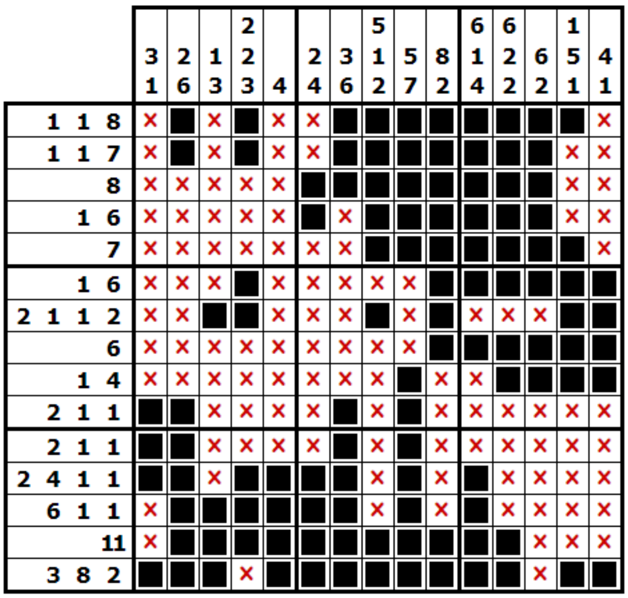
2)



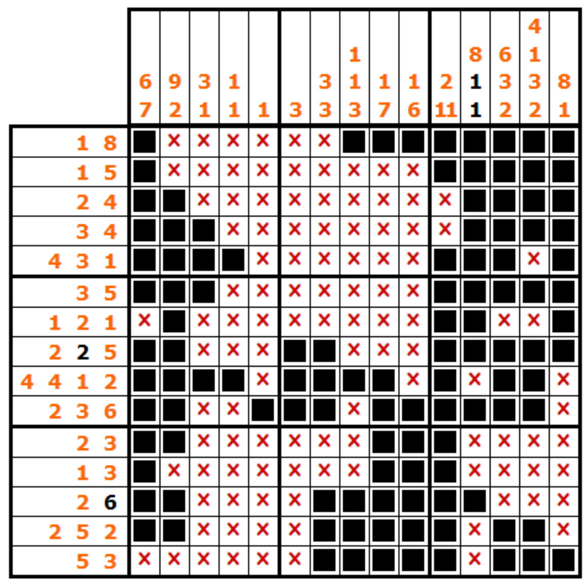
3)



4)



5)



Solve the following Kakuro puzzles[2]. Each puzzle consists of a blank grid with sum-clues in various places. The object is to fill all empty squares using numbers 1 to 9 so the sum of each horizontal block equals the clue on its left, and the sum of each vertical block equals the clue on its top. In addition, no number may be used in the same block more than once.

Example:

			6\	7\
	17\	11\4	3	1
\10	3	1	2	4
\11	5	3	1	2
\16	9	7		

6)

			23\	16\
		6\17	8	9
	3\15	2	6	7
\13	1	3	9	
\3	2	1		

7)

	17\	24\		
\15	8	7	15\	
\23	9	8	6	16\
	\17	9	1	7
		\17	8	9

8)

		27\	24\	11\
	\24	7	8	9
	15\6	3	1	2
\24	7	8	9	
\23	8	9	6	

9)

		10\	11\	4\
	\6	2	1	3
	17\7	4	2	1
\13	9	1	3	
\16	8	3	5	

10)

	16\	24\		
\17	8	9	23\	6\
\27	7	8	9	3
\16	1	7	6	2
		\9	8	1

Challenge Problems!

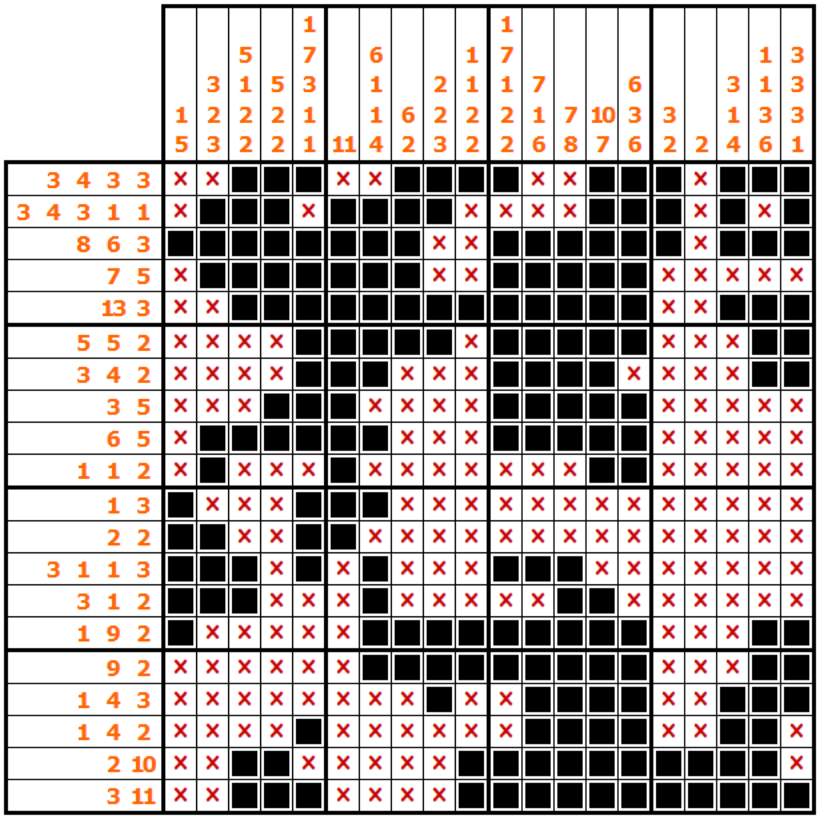
11)

		16\	13\	
	10\3	2	1	9\
\30	9	8	7	6
\11	1	5	2	3
	\4	1	3	

12)

		30\	10\	
	3\8	6	2	5\
\14	2	7	4	1
\17	1	9	3	4
	\9	8	1	

13)



Article

Saving Lives: the mathematics of tomography

: https://plus.maths.org/content/saving-lives-mathematics-tomography

Ideas to implement article

- KWL-have students create a table on their paper of 3 columns
  - K-What do students already **know** about X-rays and Imaging
  - W-What do students **want** to know
  - L-What did students **learn** from the article
- Jigsaw-break the article into pieces by paragraph and have groups read it a paragraph at a time and discuss to keep students from rushing ahead and missing important details.

Lesson 4: X-Ray Notes

X-Rays and Mathematics

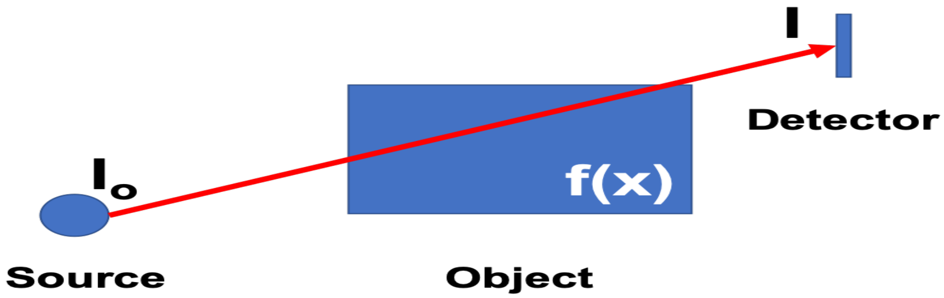
When we first think of imaging we probably think of cameras and photography. There are lots of places imaging is used: our eyes, cameras, X-Rays, CT, MRI, and Ultrasound to name a few. Even X-rays have a variety of uses. In addition to medical applications, X-rays are used for security at the airport to be able to see the inside of luggage without having to open each bag or box.

Digital image - a picture composed of pixels
Pixel - one piece of a picture. The word pixel was formed from picture + element
Array of numbers - arranging of numbers in rows and columns
Matrix - an array of numbers that is used to solve problems involving unknown quantities

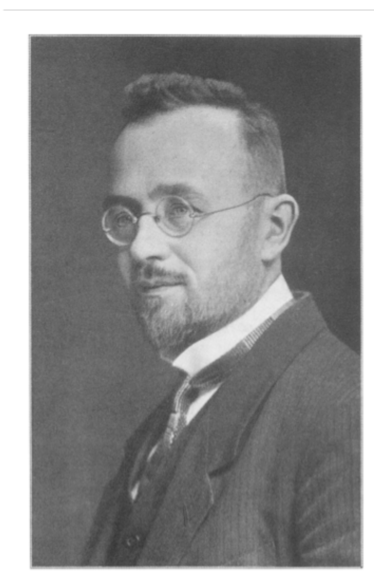
Who discovered X-rays and why are they called X-Rays?
Wilhelm Rontgen in 1895 and he called them "X" because it was an unknown type of radiation

Tomography – The word comes from Greek where "tomos" means slice or section and "grapho" means to write so it is a way to represent a slice or section of the body or other object.

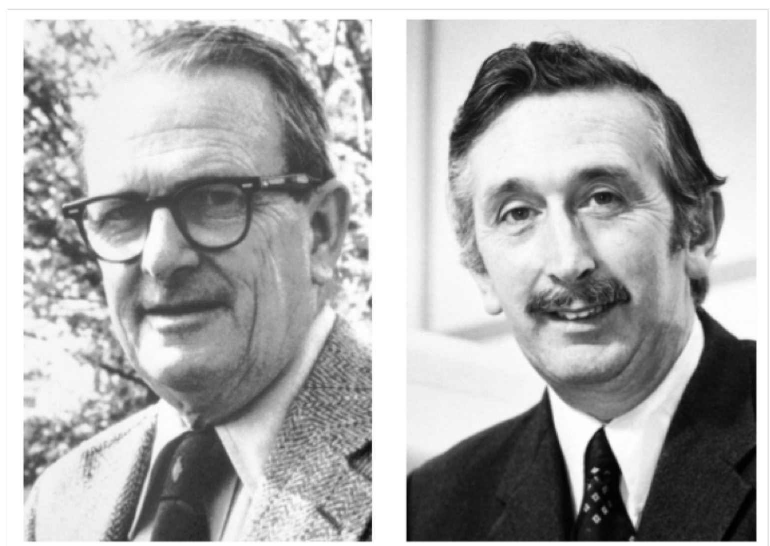
In the basic version of an X-ray machine, has a source that emits X-rays and a detector collects the X-ray as it passes through the medium of interest (e.g., human bodies) in a straight line. As the X-ray passes through the medium of interest, it encounters "resistance" and lowers in intensity. The degree to which this intensity is reduced depends on the materials it encounters along its straight-line path. Measuring this reduction in intensity can reveal the inner details of the object. Since we can't open up a person (surgery) every time we need to see inside, imaging allows us to look inside using a model similar to the one below.



Cormack and Hounsfield developed computer assisted tomography and won a joint Nobel Prize for it in 1979. Their crucial insight was that to understand the internal details, they needed to "look" at the object through multiple angles and piece together the details of the human body. This is similar, in spirit, to panorama image which also "stiches" multiple images of the same scenery. Cormack and Hounsfield independently built devices that worked on the above principles building on the mathematical insights of Radon (1887-1956).



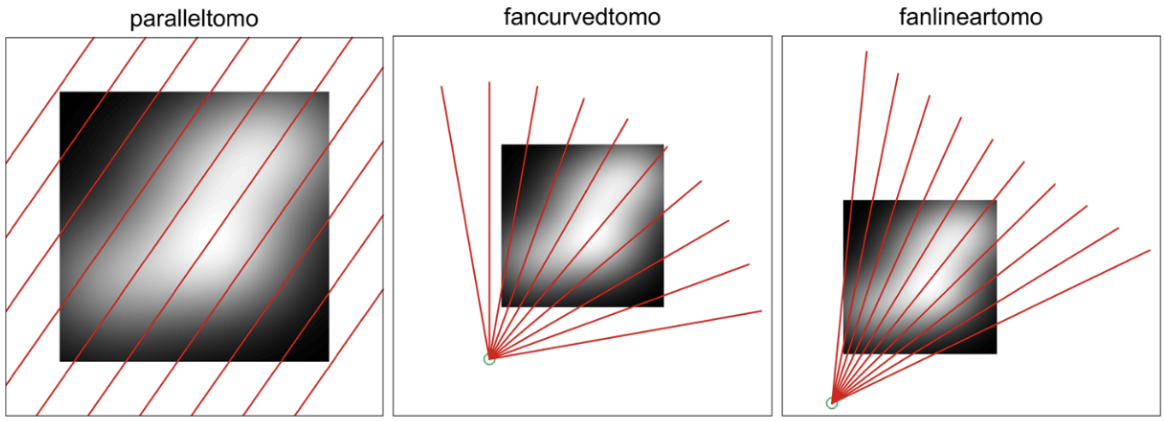
Johann Radon



Allan M. Cormack and Godfrey Hounsfield

Source: https://www.nobelprize.org/prizes/medicine/1979/summary/

The image below shows three examples of how modern X-ray devices use multiple sources and detectors to collect information about the medium. All three give a different perspective of the same image which can help determine more information about the image.



Source: Hansen, Jorgensen. Numerical Algorithms 79.1 (2018): 107-137.

Games related to X-ray imaging

You may have heard of a Sudoku, but in this class we will solve two puzzles related to Sudoku called Kakuro and Nonograms.

How are these puzzles related to x-ray imaging?

When a computer algorithm is generating or reconstructing images using x-ray data, it is trying to solve a puzzle very similar to that a Nonogram or Kakuro. The key similarities are:

- We do not know the internal details (the variables are the values inside).
- We are given some indirect details on how the numbers add up along rows and columns.

However, the rules of the games (Kakuro, nonograms) are designed in such a way that each puzzle, however hard it may seem, has a solution that is unique. Unfortunately, that is where the similarity to reconstructing x-ray images ends. To explain these challenges, first consider the notion of a well-posed problem.

For a problem to be **well-posed**:

A solution exists
The solution is unique
The solution's behavior changes continuously with the inputs

If a problem is not well-posed it is called **ill-posed**.

Unlike puzzles/games what are well-posed, imaging problems tend to be ill-posed, which makes them a lot harder to solve in practice. Two other reasons make x-ray imaging more complicated:

1. The sources and detectors are not perfect, they collect noisy measurements. You may have seen a similar effect when you take photographs in your cellphone cameras in low-light settings.
2. Patients tend to move during imaging. This means in between two sets of x-ray measurements the X-ray scanner is looking at two different "patients."

These uncertainties make X-ray imaging more complicated but also more interesting. More information about X-ray imaging can be found on this website: https://www.whydomath.org/node/tomography/index.html.

To show the complications involving ill-posed problems, we will play a couple of simple ill-posed games.

Milk Delivery

In the article below milk delivery is discussed starting in paragraph 5. This is an option activity that teachers could demonstrate to help visual and kinesthetic learners understand the ambiguity of X-Ray imaging. https://plus.maths.org/content/saving-lives-mathematics-tomography

Guess My Square

Directions: Fill in the square that is labeled “your square” with digits between 1 and 9. Then sum the columns and rows of your square. Tell the opposing team the sums of the columns and rows only. The opposing team now needs to fill in your square with digits between 1 and 9 to try to guess the numbers that you filled your square in with. You will also try to guess your opposing team’s digits. You will put your guess into the square labeled “their square”.

Your Square

Their Square

***Note to teacher: There are infinite solutions to these squares, and you will want the students to realize that through this activity.**

Lesson 5: Solving ill-posed puzzles

Standard	
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices. DCS.N.2.2 Interpret solutions found using matrix operations in context	**Topic/Day:** Finding solutions to ill posed problems **Content Objective:** Use the process of rays in an X-Ray to solve equations **Vocabulary:** Gaussian Elimination, Matrix, Vector **Materials Needed:** Guided Notes, Student Worksheet **~85 minutes**

	Time	Student Does	Teacher Does
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~ 10 mins	Students should work on answering the posed questions about Gaussian Elimination as review.	Encourage students to refer back to their notes about Gaussian Elimination to answer the posed questions.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~30 mins	Students will complete the Lesson 5 - guided notes about X-Rays and Linear Algebra	Gives guided notes and walk them through. Be sure to let them try some of the notes on their own.

Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~30 mins	Students will complete the Lesson 5 - student worksheet applying what they learned in the notes to a new scenario.	Actively monitor the classroom to answer questions and encourage collaboration between students.
Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	~ 15 mins	Have the students complete the Lesson 5 – Exit Ticket where they will create their own diagram and explain why or why not they think it would result in a unique solution. Have some of the groups share their image and explanation with the rest of the class.	Provide the students with the exit ticket. Actively monitor the students as they are working on the exit ticket. Choose groupings to present based on their image and explanation – you will want to choose groups that will add to the knowledge and understanding of students who may not fully understand.

Warm Up-Key

X-Ray Day 2 – Warm Up/Engage Name: ______________________________

Answer Key Date: ________________ Period: ________

What are the 3 Elementary Row Operations that you can use while completing Gaussian Elimination?

1) **Exchanging two rows**

2) **Multiplying a row by a nonzero scalar**

3) **Adding a multiple of one row to another**

What are the 3 possible conclusions that you can make about solving a system of equations?

1) **There exists one unique solution**

2) There is no solution

3) There is an infinite number of solutions

During Gaussian Elimination, how do you determine which case you have ended up with?

1) You end up with 1s on the main diagonal of the first square portion of the matrix

2) You end with leading 0s on the bottom row and a constant in the last column of the bottom row/or any other row (e.g., [0 0 … 0 | 5])

3) Most often, you end up with all zeros on the bottom row/another row (e.g., [0 0 … 0 | 0])

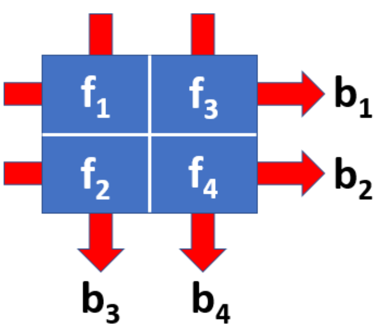
Guided Notes Key

X-Rays – Day 2 Guided Notes
Answer Key

Name: ______________________________
Date: __________________ Period: _______

In the previous lesson, we played the "Guess my Square" game where we discovered that there are more than one solution sets to our squares. Today we are going to look at how we can use a similar approach and result in one, unique solution. This approach that we will look at today is how the rays in x-rays move through the body to result in the best image outcome possible.

Given the information in the diagram below, write the system of equations that corresponds.



$$\begin{aligned} f_1 \quad\quad\quad + f_3 \quad\quad\quad &= b_1 \\ f_2 \quad\quad\quad + f_4 &= b_2 \\ f_1 + f_2 \quad\quad\quad\quad\quad\quad &= b_3 \\ f_3 + f_4 &= b_4 \end{aligned}$$

If we think about the above image as matrix x and $\boldsymbol{x} = \begin{bmatrix} f_1 & f_3 \\ f_2 & f_4 \end{bmatrix}$, the first thing we want to do it vectorize this matrix.

$$\boldsymbol{x} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

Now, write the matrix-vector equation in the form *Ax=b.*

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Consider the true solution of $\boldsymbol{x} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. What does this matrix look like if we vectorize it?

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Now, write down the corresponding vector *b*

$$\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 3 \\ 7 \end{bmatrix}$$

Using vector *b* and matrix *A* from above, write the augmented matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 1 & 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 1 & | & 7 \end{bmatrix}$$

Now we solve for vector *x* by using Gaussian Elimination.

1) $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 1 & 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 1 & | & 7 \end{bmatrix} \leftarrow -R_1+R_3$$

2) $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 1 & -1 & 0 & | & -1 \\ 0 & 0 & 1 & 1 & | & 7 \end{bmatrix} \leftarrow R_3-R_2$$

3) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & -1 & -1 & | & -7 \\ 0 & 0 & 1 & 1 & | & 7 \end{bmatrix}$ ← -R3

4) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 1 & 1 & | & 7 \end{bmatrix}$ ← R4-R3

5) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

Set up the new system of equations from your final matrix in your Gaussian Elimination.

$$\begin{array}{ccccc} f_1 & & +f_3 & & = 4 \\ & f_2 & & +f_4 & = 6 \\ & & f_3 & +f_4 & = 7 \\ & & & 0 & = 0 \end{array}$$

What does the $\mathbf{0} = \mathbf{0}$ tell you about the system of equations?
There is no unique solution to this system of equations.
Therefore, according to our rules of Gaussian Elimination, we will end up with infinite solutions.

Since you are not able to solve for a unique solution, instead solve for $f_1, f_2, and\ f_3$ in terms of f_4.
Solve for f_3:

$$\begin{array}{ccc} f_3 & +f_4 & = 7 \\ & -f_4 & -f_4 \\ f_3 & = 7 & -f_4 \end{array}$$

Solve for f_2:

$$\begin{array}{ccc} f_2 & +f_4 & = 6 \\ & -f_4 & -f_4 \\ f_2 & = 6 & -f_4 \end{array}$$

Solve for f_1:

$$\begin{array}{ccc} f_1 & +f_3 & = 4 \\ f_1 & +(7 - f_4) & = 4 \end{array}$$

$$-(7-f_4) \qquad -(7-f_4)$$

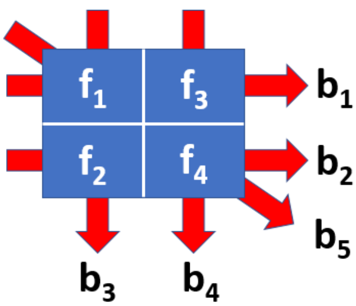
$$f_1 \qquad = 4 \qquad -7+f_4$$

$$f_1 \qquad = -3 \qquad +f_4$$

Refer to the "Guess my Square" game from yesterday. What value of f_4 would make your team guess the opposing team's answer?
Answer will vary

Making the solution unique

Since the above system gives you infinite solutions, we must now add in more rays. Given the information in the new diagram below, write the new system of equations that corresponds.



$$\begin{aligned}
f_1 \quad\quad\quad &+f_3 \quad\quad\quad &= b_1 \\
f_2 \quad\quad\quad &\quad\quad +f_4 &= b_2 \\
f_1 \quad +f_2 \quad &\quad\quad\quad &= b_3 \\
&\quad f_3 \quad +f_4 &= b_4 \\
f_1 \quad\quad\quad &\quad\quad +f_4 &= b_5
\end{aligned}$$

Write the matrix-vector equation in the form *Ax=b.*

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

If matrix $x = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, write vector *x.*

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Now, write down the corresponding vector *b*.

$$\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 3 \\ 7 \\ 5 \end{bmatrix}$$

Using the new vector *b* and new matrix *A* from above, write the augmented matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 1 & 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 1 & | & 7 \\ 1 & 0 & 0 & 1 & | & 5 \end{bmatrix}$$

Now prove that there is a unique solution and solve for vector *x* by using Gaussian Elimination.

1) $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 1 & 1 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 1 & | & 7 \\ 1 & 0 & 0 & 1 & | & 5 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_4 \leftrightarrow R_3 \\ \\ \\ \end{matrix}$$

2) $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 1 & 1 & 0 & 0 & | & 3 \\ 1 & 0 & 0 & 1 & | & 5 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_1 - R_4 \\ \\ \\ \end{matrix}$$

3) $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & -1 & 1 & 0 & | & 1 \\ 1 & 0 & 0 & 1 & | & 5 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_2 + R_4 \\ \\ \\ \end{matrix}$$

4) $$\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 1 & 1 & | & 7 \\ 1 & 0 & 0 & 1 & | & 5 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_3 - R_4 \\ \\ \\ \end{matrix}$$

5) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & 0 & | & 0 \\ 1 & 0 & 0 & 1 & | & 5 \end{bmatrix} \leftarrow R_5 \leftarrow\rightarrow R_4$

6) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 1 & 0 & 0 & 1 & | & 5 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \leftarrow R_1\text{-}R_4$

7) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 1 & -1 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \leftarrow R_3\text{-}R_4$

8) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & 2 & | & 8 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \leftarrow \tfrac{1}{2} R_4$

9) $\begin{bmatrix} 1 & 0 & 1 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & | & 6 \\ 0 & 0 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & 1 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

Write the new system of equations from the final matrix in your Gaussian Elimination.

$$\begin{array}{ccccc} f_1 & & +f_3 & & = 4 \\ & f_2 & & +f_4 & = 6 \\ & & f_3 & +f_4 & = 7 \\ & & & f_4 & = 4 \\ & & & 0 & = 0 \end{array}$$

Are you able to solve this system of equations? Why?

Yes, because we have 4 equations with 4 variables to solve for, so the fifth equation of 0=0 doesn't affect us trying to solve for a unique solution.

If yes, solve the system of equations.

Solve for f_4:

$$f_4 = 4$$

Solve for f_3:

$$\begin{array}{ccc} f_3 & +4 & = 7 \\ & -4 & -4 \\ f_3 & & = 3 \end{array}$$

Solve for f_2:

$$\begin{array}{ccc} f_2 & +4 & = 6 \\ & -4 & -4 \\ f_2 & & = 2 \end{array}$$

Solve for f_1:

$$\begin{array}{ccc} f_1 & +3 & = 4 \\ & -3 & -3 \\ f_1 & & = 1 \end{array}$$

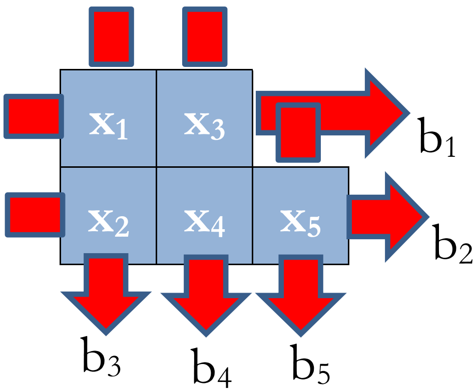
☼ **Final Answer** $x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **(Note: You should get the same vector *x* that we have above)**

Worksheet Key

X-Rays Student Worksheet
Answer Key

Name: ___________________________________
Date: _____________________ Period: ________

Given the information in the diagram below, write the system of equations that corresponds.



$$
\begin{array}{cccccl}
x_1 & & +x_3 & & & = b_1 \\
& x_2 & & +x_4 & +x_5 & = b_2 \\
x_1 & +x_2 & & & & = b_3 \\
& & x_3 & +x_4 & & = b_4 \\
& & & & x_5 & = b_5
\end{array}
$$

Write the matrix-vector equation in the form *Ax=b*.

$$
\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}
$$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, write down the corresponding vector *b*.

$$
b = \begin{bmatrix} 4 \\ 11 \\ 3 \\ 7 \\ 5 \end{bmatrix}
$$

Using vector *b* and matrix *A* from above, write the augmented matrix.

$$
\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right]
$$

Now prove that there is a unique solution and solve for vector *x* by using Gaussian Elimination.

$$1)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_4 \leftrightarrow R_3$$

$$2)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_1 - R_4$$

$$3)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_2 + R_4$$

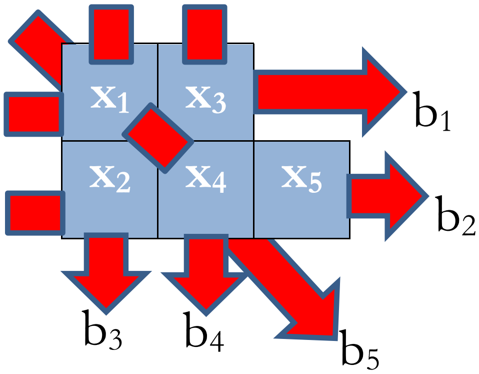
$$4)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 1 & 12 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_3 - R_4$$

$$5)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right] \leftarrow -R_4$$

$$6)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right] \leftarrow R_4 - R_5$$

$$7)\quad \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

As you can see, this will give you infinite solutions. Why did this system give you infinite solutions?
When the last row of the matrix is all zeros, that means that you will have infinite solutions.
Since the above system gives you infinite solutions, we must now add in more rays. Given the information in the new diagram below, write the new system of equations that corresponds.



$$\begin{array}{llllll} x_1 & & +x_3 & & & = b_1 \\ & x_2 & & +x_4 & +x_5 & = b_2 \\ x_1 & +x_2 & & & & = b_3 \\ & & x_3 & +x_4 & & = b_4 \\ x_1 & & & +x_4 & & = b_5 \end{array}$$

Write the matrix-vector equation in the form *Ax=b.*

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, write down the corresponding vector *b.*

$$b = \begin{bmatrix} 4 \\ 11 \\ 3 \\ 7 \\ 5 \end{bmatrix}$$

Using the new vector *b* and new matrix *A* from above, write the augmented matrix.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 1 & 11 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 7 \\ 1 & 0 & 0 & 1 & 0 & 5 \end{array}\right]$$

Now prove that there is a unique solution and solve for vector *x* by using Gaussian Elimination.

1) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 1 & 1 & 0 & 0 & 0 & | & 3 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 1 & 0 & 0 & 1 & 0 & | & 5 \end{bmatrix} \leftarrow R_4 \leftrightarrow R_3$

2) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 1 & 1 & 0 & 0 & 0 & | & 3 \\ 1 & 0 & 0 & 1 & 0 & | & 5 \end{bmatrix} \leftarrow R_1 - R_4$

3) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 0 & -1 & 1 & 0 & 0 & | & 1 \\ 1 & 0 & 0 & 1 & 0 & | & 5 \end{bmatrix} \leftarrow R_2 + R_4$

4) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & 1 & 1 & | & 12 \\ 1 & 0 & 0 & 1 & 0 & | & 5 \end{bmatrix} \leftarrow R_3 - R_4$

5) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 0 & 0 & 0 & 0 & -1 & | & -5 \\ 1 & 0 & 0 & 1 & 0 & | & 5 \end{bmatrix} \leftarrow R_5 \leftrightarrow -R_4$

6) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 1 & 0 & 0 & 1 & 0 & | & 5 \\ 0 & 0 & 0 & 0 & 1 & | & 5 \end{bmatrix} \leftarrow R_1 - R_4$

7) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 0 & 0 & 1 & -1 & 0 & | & -1 \\ 0 & 0 & 0 & 0 & 1 & | & 5 \end{bmatrix} \leftarrow R_3 - R_4$

8) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 0 & 0 & 0 & 2 & 0 & | & 8 \\ 0 & 0 & 0 & 0 & 1 & | & 5 \end{bmatrix}$ ← ½ R_4

9) $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & 1 & 1 & | & 11 \\ 0 & 0 & 1 & 1 & 0 & | & 7 \\ 0 & 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & 0 & 0 & 1 & | & 5 \end{bmatrix}$

Write and solve the new system of equations from the final matrix in your Gaussian Elimination.

$$\begin{array}{cccccc} x_1 & & +x_3 & & & = 4 \\ & x_2 & & +x_4 & +x_5 & = 11 \\ & & x_3 & +x_4 & & = 7 \\ & & & x_4 & & = 4 \\ & & & & x_5 & = 5 \end{array}$$

Solve for x_3:

$$\begin{array}{ccc} x_3 & +4 & = 7 \\ & -4 & -4 \\ x_3 & & = 3 \end{array}$$

Solve for x_2:

$$\begin{array}{cccc} x_2 & +4 & +5 & = 11 \\ x_2 & +9 & = 11 & \\ & -9 & -9 & \\ x_2 & & & = 2 \end{array}$$

Solve for x_1:

$$\begin{array}{ccc} x_1 & +3 & = 4 \\ & -3 & -3 \\ x_1 & & = 1 \end{array}$$

☼ **Final Answer:** $x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ **(Note: You should get the same vector *x* that we have above)**

Exit Ticket

X-Rays Day 2 – Exit Ticket

Name: ___________________________________

Date: ___________________ Period: __________

Create your own square image and rays below (this image must be different from one used in class today):

Will this image produce a unique solution? Why?

References and Additional Readings

- Bartkovich, K. G., Goebel, J. A., Graves, J. L., Teague, D. J., Barrett, G. B., Compton, H. L., & Whitehead, K. (2000). *Contemporary Precalculus through Applications*. New York: Glencoe/McGraw-Hill

- How Math Can Save Your Life: Tomography. (n.d.). Retrieved July 23, 2020, from https://www.whydomath.org/node/tomography/index.html

- Saving lives: The mathematics of tomography. (2018, July 26). Retrieved July 23, 2020, from https://plus.maths.org/content/saving-lives-mathematics-tomography

- Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

Appendices Lesson Materials – Student Versions

Lesson 1 - Guided Notes - Student Version

- Student version begins on next page

Matrix Addition, Subtraction and Scalar Multiplication

A university is taking inventory of the books they carry at their two biggest bookstores.
The East Campus bookstore carries the following books:

Hardcover: Textbooks-5280; Fiction-1680; NonFiction-2320; Reference-1890

Paperback: Textbooks-1930; Fiction-2705; NonFiction-1560; Reference-2130

The West Campus bookstore carries the following books:

Hardcover: Textbooks-7230; Fiction-2450; NonFiction-3100; Reference-1380

Paperback: Textbooks-1740; Fiction-2420; NonFiction-1750; Reference-1170

In order to work with this information, we can represent the inventory of each bookstore using an organized array of numbers known as a *matrix*.

Definitions: A __________ is a rectangular table of entries and is used to organize data in a way that can be used to solve problems. The following is a list of terms used to describe matrices:

- A matrix's __________________________ is written by listing the number of rows "by" the number of columns.
- The values in a matrix, A, are referred to as _____________ or _____________. The entry in the "m^{th}" row and "n^{th}" column is written as a_{mn}.
- A matrix is _____________ if it has the same number of rows as it has columns.
- If a matrix has only one row, then it is a row ___________. If it has only one column, then the matrix is a column _____________.
- The _________________ of a matrix, A, written A^T, switches the rows with the columns of A and the columns with the rows.
- Two matrices are ____________ if they have the same size and the same corresponding entries.

The inventory of the books at the East Campus bookstore can be represented with the following **2 x 4** matrix:

$$E = \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Similarly, the West Campus bookstore's inventory can be represented with the following matrix:

$$W = \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Adding and Subtracting Matrices

In order to add or subtract matrices, they must first be of the same ________________________. The result of the addition or subtraction is a matrix of the same size as the matrices themselves, and the entries are obtained by adding or subtracting the elements in corresponding positions.

In our campus bookstores example, we can find the total inventory between the two bookstores as follows:

$$E + W = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} + \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Question: *Is matrix addition commutative* $(e.g., A + B = B + A)$*? Why or why not?*

Question: *Is matrix subtraction commutative* $(e.g., A - B = B - A)$*? Why or why not?*

Question: *Is matrix addition associative* $(e.g., (A + B) + C = A + (B + C))$*? Why or why not?*

Question: *Is matrix subtraction associative* $(e.g., (A - B) - C = A - (B - C))$*? Why or why not?*

Scalar Multiplication

Multiplying a matrix by a constant (or *scalar*) is as simple as multiplying each entry by that number! Suppose the bookstore manager in East Campus wants to double his inventory. He can find the number of books of each type that he would need by simply multiplying the matrix E by the scalar (or constant) 2. The result is as follows:

$$2E = 2 * \begin{bmatrix} & & & \\ & & & \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} & & & \\ & & & \end{bmatrix} \end{array}$$

Exercises: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed:

b. $A + B$

b. $B - A$

$c.$ $A - C$

d. $C - A$

e. $5B$

f. $-A + 4C$

g. $B - D$

h. $2C - 6A$

i. $B^T + D$

Lesson 2 - Guided Notes – Student Version

- ❖ Student version begins on next page

Matrix Multiplication

The Metropolitan Opera is planning its last cross-country tour. It plans to perform *Carmen* and *La Traviata* in Atlanta in May. The person in charge of logistics wants to make plane reservations for the two troupes. *Carmen* has 2 stars, 25 other adults, 5 children, and 5 staff members. *La Traviata* has 3 stars, 15 other adults, and 4 staff members. There are 3 airlines to choose from. Redwing charges round-trip fares to Atlanta of $630 for first class, $420 for coach, and $250 for youth. Southeastern charges $650 for first class, $350 for coach, and $275 for youth. Air Atlanta charges $700 for first class, $370 for coach, and $150 for youth. Assume stars travel first class, other adults and staff travel coach, and children travel for the youth fare.

Use multiplication and addition to find the total cost for each troupe to travel each of the airlines.

It turns out that we can solve problems like these using a matrix operation, specifically **matrix multiplication**!

We first note that matrix multiplication is only defined for matrices of certain sizes. For the product AB of matrices A and B, where A is an $m\ x\ n$ matrix, B must have the same number of rows as A has columns. So, B must have size _____ $x\ p$. The product AB will have size _____________.

Exercises

The following is a set of abstract matrices (without row and column labels):

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad O = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 5 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

List at least 5 orders of pairs of matrices from this set for which the product is defined. State the dimension of each product.

Back to the opera…

Define two matrices that organize the information given:

$$\begin{array}{rc} & \begin{array}{ccc}\textbf{stars} & \textbf{adults} & \textbf{children}\end{array} \\ \begin{array}{r}\textit{Carmen} \\ \textit{La Traviata}\end{array} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array} \qquad \begin{array}{rc} & \begin{array}{ccc}\textbf{Red} & \textbf{South} & \textbf{Air}\end{array} \\ \begin{array}{r}\textbf{stars} \\ \textbf{adults} \\ \textbf{children}\end{array} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array}$$

We can multiply these two matrices to obtain the same answers we obtained above, all in one matrix!

$$\begin{array}{rc} & \begin{array}{ccc}\textbf{stars} & \textbf{adults} & \textbf{children}\end{array} \\ \begin{array}{r}\textit{Carmen} \\ \textit{La Traviata}\end{array} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array} \cdot \begin{array}{rc} & \begin{array}{ccc}\textbf{Red} & \textbf{South} & \textbf{Air}\end{array} \\ \begin{array}{r}\textbf{stars} \\ \textbf{adults} \\ \textbf{children}\end{array} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array}$$

$$= \begin{array}{rc} & \begin{array}{ccc}\textbf{Red} & \textbf{South} & \textbf{Air}\end{array} \\ \begin{array}{r}\textit{Carmen} \\ \textit{La Traviata}\end{array} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array}$$

Carmen/Redwing:

Carmen/Southeastern:

Carmen/Air Atlanta:

La Traviata/Redwing:

La Traviata/Southeastern:

La Traviata/Air Atlanta:

Exercises[2]

3. The K.L. Mutton Company has investments in three states - North Carolina, North Dakota, and New Mexico. Its deposits in each state are divided among bonds, mortgages, and consumer loans. The amount of money (in millions of dollars) invested in each category on June 1 is displayed in the table below.

	NC	ND	NM
Bonds	13	25	22
Mort.	6	9	4
Loans	29	17	13

The current yields on these investments are 7.5% for bonds, 11.25% for mortgages, and 6% for consumer loans. Use matrix multiplication to find the total earnings for each state.

4. Several years ago, Ms. Allen invested in growth stocks, which she hoped would increase in value over time. She bought 100 shares of stock A, 200 shares of stock B, and 150 shares of stock C. At the end of each year she records the value of each stock. The table below shows the price per share (in dollars) of stocks A, B, and C at the end of the years 1984, 1985, and 1986.

	1984	1985	1986
Stock A	68.00	72.00	75.00
Stock B	55.00	60.00	67.50
Stock C	82.50	84.00	87.00

Calculate the total value of Ms. Allen's stocks at the end of each year.

[2] Adapted from Bartkovich, K. G., Goebel, J. A., Graves, J. L., Teague, D. J., Barrett, G. B., Compton, H. L., ... & Whitehead, K. (2000). *Contemporary Precalculus through Applications*. New York: Glencoe/McGraw-Hill

3. The Sound Company produces stereos. Their inventory includes four models - the Budget, the Economy, the Executive, and the President models. The Budget needs 50 transistors, 30 capacitors, 7 connectors, and 3 dials. The Economy model needs 65 transistors, 50 capacitors, 9 connectors, and 4 dials. The Executive model needs 85 transistors, 42 capacitors, 10 connectors, and 6 dials. The President model needs 85 transistors, 42 capacitors, 10 connectors, and 12 dials. The daily manufacturing goal in a normal quarter is 10 Budget, 12 Economy, 11 Executive, and 7 President stereos.

   a. How many transistors are needed each day? Capacitors? Connectors? Dials?

   b. During August and September, production is increased by 40%. How many Budget, Economy, Executive, and President models are produced daily during these months?

   c. It takes 5 person-hours to produce the Budget model, 7 person-hours to produce the Economy model, 6 person-hours for the Executive model, and 7 person-hours for the President model. Determine the number of employees needed to maintain the normal production schedule, assuming everyone works an average of 7 hours each day. How many employees are needed in August and September?

4. The president of the Lucrative Bank is hoping for a 21% increase in checking accounts, a 35% increase in savings accounts, and a 52% increase in market accounts. The current statistics on the number of accounts at each branch are as follows:

$$\begin{array}{r c} & \begin{array}{ccc} \textit{Checking} & \textit{Savings} & \textit{Market} \end{array} \\ \begin{array}{r} \textit{Northgate} \\ \textit{Downtown} \\ \textit{South Square} \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array}$$

What is the goal for each branch in each type of account? (HINT: multiply by a 3×2 matrix with certain nonzero entries on the diagonal and zero entries elsewhere.) What will be the total number of accounts at each branch?

Lesson 3 - Guided Notes - Student Version

- Student version begins on next page

Solving Linear Systems of Equations Using Gaussian Elimination

A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling.

How can we represent this problem with a system of equations?

Let $x =$

Let $y =$

Let $z =$

We therefore have the following system of equations:

Example: Consider the following system of linear equations (recall this from Algebra II):

$$\begin{aligned} x + 3y &= 0 \\ x + y + z &= 1 \\ 3x - y - z &= 11 \end{aligned}$$

We can solve this system by representing it using matrices.

We will name the ________________ matrix $A = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$, the **variable vector** $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the **column vector** $B = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$. So, our **matrix equation** (also referred to as a linear system of equations) representing the system can be written as $AX = B$:

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

One way to solve this system is to use an approach known as

____________________________________, or **row reduction.**

Gaussian Elimination

You may recall from your prior mathematics work that there are three possible conclusions we can make about the solution to a system of equations.

Case 1: There exists one unique solution.
Case 2: There is no solution.
Case 3: There is an infinite number of solutions.

<u>Case 1</u>: There exists one unique solution.

Recall our example from above:

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

To begin, we write the associated ______________________________, which is written in the following form:

$$\left[\begin{array}{c|c} \quad\quad\quad\quad\quad\quad & \quad\quad \\ \\ \\ \\ \end{array}\right]$$

To apply the method on a matrix, we use ____ to

modify the matrix. Our goal is to end up with the __________________________, which is an $n\ x\ n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, on the left side of the augmented matrix.

Our solution to the system of equations will be the resulting matrix on the right side of the augmented matrix. This is because the resulting augmented matrix would represent a system of equations in which each variable could be solved for (if a solution exists).

Elementary Row Operations:

There are three operations that can be applied to modify the matrix and still preserve the solution to the system of equations.

- Exchanging two rows (which represents the switching the listing order of two equations in the system)
- Multiplying a row by a nonzero scalar (which represents multiplying both sides of one of the equations by a nonzero scalar)
- Adding a multiple of one row to another (which represents does not affect the solution, since both equations are in the system)

For our example…

$$\begin{aligned} x + 3y &= 0 && R_1 \\ x + y + z &= 1 && R_2 \\ 3x - y - z &= 11 && R_3 \end{aligned}$$

System of equations	Row operation	Augmented matrix

Back to our opening problem! A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling. How much money will each location receive in grants?

Rewrite your system of equations from earlier in this lesson:

We can represent this system using the following linear systems of equations:

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$$

The augmented matrix for this system is:

Using elementary row operations, we find that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

So, ____________________ goes to the East location, __________________ goes to the West location,

and ____________________ goes to the South location.

<u>Case 2</u>: There is no solution.

Consider the system of equations:

$$\begin{aligned} 2x - y + z &= 1 \\ 3x + 2y - 4z &= 4 \\ -6x + 3y - 3z &= 2 \end{aligned}$$

Augmented matrix:

Using row operation $R_3 + 3R_1 \rightarrow R_3$, we get

We note that the third row in the augmented matrix is a false statement, so there is no solution to this system.

<u>Case 3</u>**: There is an infinite number of solutions.**

Consider the system of equations:

$$\begin{gathered} x - y + 2z = -3 \\ 4x + 4y - 2z = 1 \\ -2x + 2y - 4z = 6 \end{gathered}$$

Augmented matrix: $\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array}\right]$

Using row operations $R_2 - 4R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$, we get

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array}\right]$$

This represents a system that leaves us with 2 equations and 3 unknowns. So, we are unable to solve for one variable without expressing it in terms of another. This gives us an infinite number of solutions.

Exercises

For each of the following problems, identify your variables and write a system of equations to represent the problem. Then use Gaussian elimination to solve the system.

1. The Frodo Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat is $42 and $30 per acre, respectively. Mr. Frodo has $18,600 available for cultivating these crops. If he wants to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant? (Adapted from *Finite Mathematics,* Tan p. 93 #51[3])

2. The Coffee Cart sells a blend made with two different coffees, one costing $2.50 per pound, and the other costing $3.00 per pound. If the blended coffee sells for $2.80 per pound, how much of each coffee is used to obtain the blend? (Assume that the weight of the coffee blend is 100 pounds.) (Adapted from *Finite Mathematics,* Tan p. 93 #53)

[3] Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

3. The Maple Movie Theater has a seating capacity of 900 and charges \$2 for children, \$3 for students, and \$4 for adults. At a screening with full attendance last week, there were half as many adults as children and students combined. The receipts totaled \$2800. How many adults attended the show? (Adapted from *Finite Mathematics,* Tan p. 97 #60)

4. The Toolies have a total of \$100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12% per year, while bonds pay 8% per year, and the money market account pays 4% per year. They have decided that the amount invested in stocks should be equal to the difference between the amount invested in bonds and 3 times the amount invested in the money market account. How should the Toolies allocate their resources if they require an annual income of \$10,000 from their investments? (Adapted from *Finite Mathematics,* Tan p. 106 #36)

Lesson 4 - Games – Student Version

- Student version begins on next page

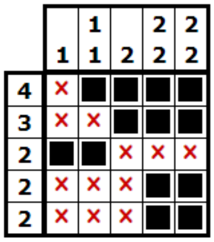
Nonogram and Kakuro Puzzles

Name: ______________________________

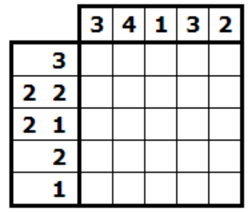
Date: ________________ Period: _______

Solve the following Nonogram puzzles[1]. Nonograms are picture logic puzzles in which cells in a grid must be colored or left blank according to the numbers at the side of the grid to reveal a hidden picture. For example: 1 5 2 means 1 square, 5 squares, and 2 squares, in this order, separated by one or more spaces between them.

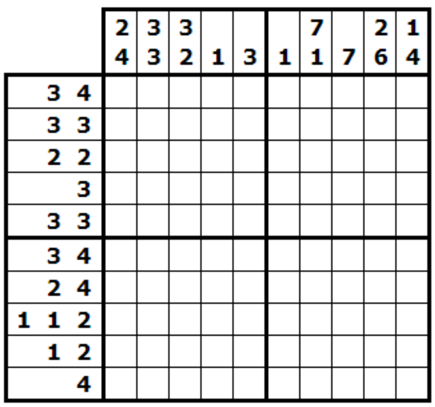
Example:



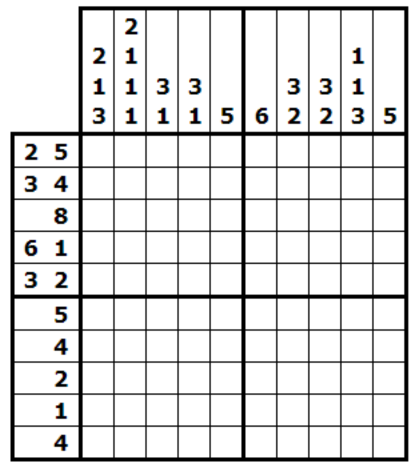
1)



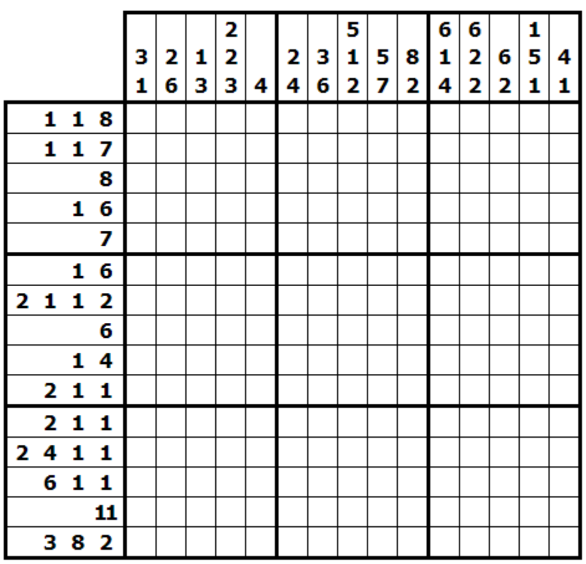
2)



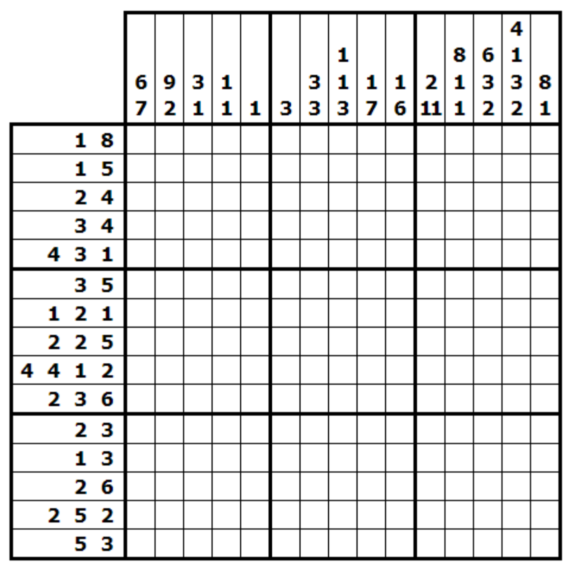
3)



4)

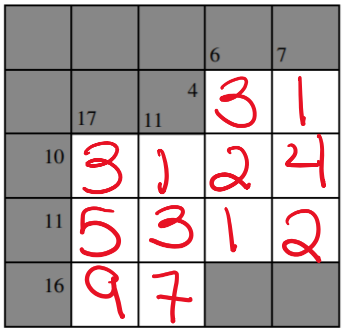


5)



Solve the following Kakuro puzzles[2]. Each puzzle consists of a blank grid with sum-clues in various places. The object is to fill all empty squares using numbers 1 to 9 so the sum of each horizontal block equals the clue on its left, and the sum of each vertical block equals the clue on its top. In addition, no number may be used in the same block more than once.

Example:



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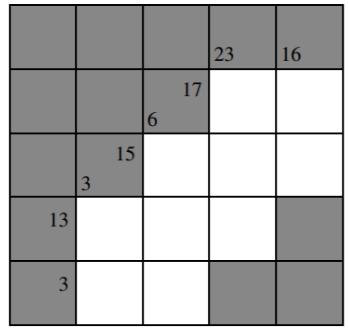
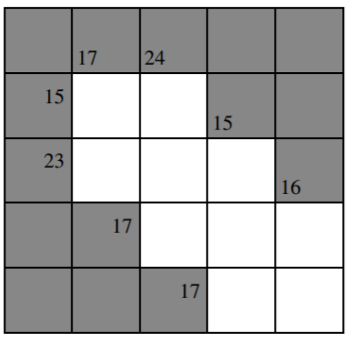
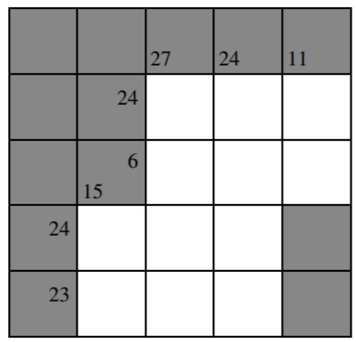


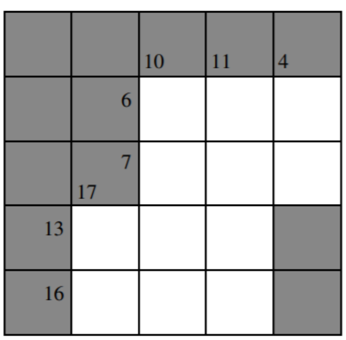
7)



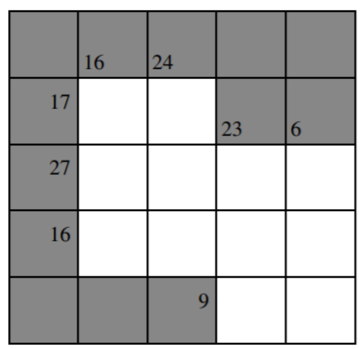
8)



9)

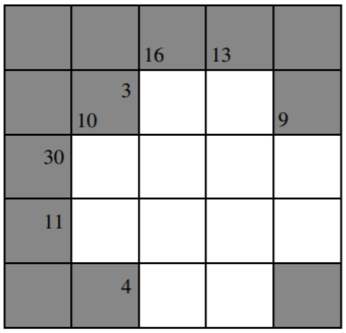


10)

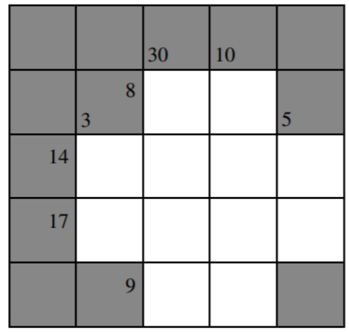


Challenge Problems!

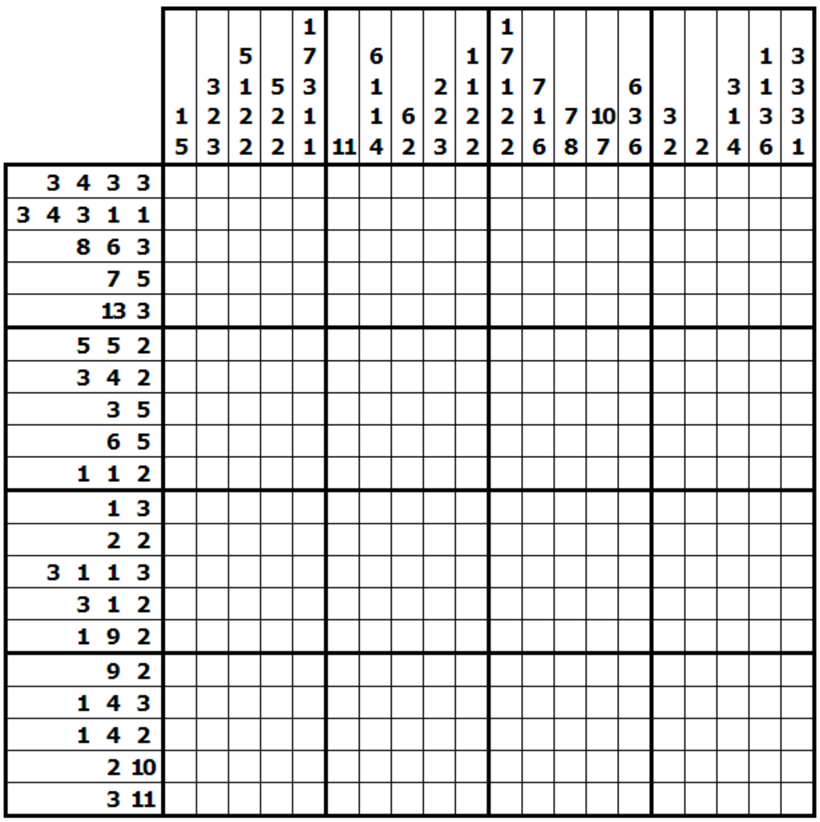
11)



12)



13)



Lesson 5-Warm Up-Student Version

- Student version begins on next page

X-Ray Day 2 – Warm Up/Engage

Name: ______________________________

Date: ____________________ Period: ________

What are the 3 Elementary Row Operations that you can use while completing Gaussian Elimination?

4)

5)

6)

What are the 3 possible conclusions that you can make about solving a system of equations?

-
-
-

During Gaussian Elimination, how do you determine which case you have ended up with?

4)

5)

6)

Lesson 5 – Guided Notes Student Version

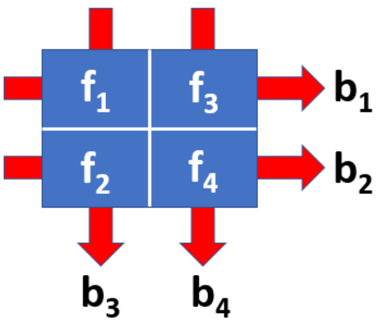
- Student version begins on next page

X-Rays – Day 2 Guided Notes Name: ___________________________________

Date: _____________________ Period: _________

Yesterday we played the "Guess my Square" game where we discovered that there are more than one solution sets to our squares. Today we are going to look at how we can use a similar approach and result in one, unique solution. This approach that we will look at today is how the rays in x-rays move through the body to result in the best image outcome possible.

Given the information in the diagram below, write the system of equations that corresponds.



If we think about the above image as matrix x and $x = \begin{bmatrix} f_1 & f_3 \\ f_2 & f_4 \end{bmatrix}$, the first thing we want to do it vectorize this matrix.

$$x = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Now, write the matrix-vector equation in the form *Ax=b.*

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Consider the true solution of $x = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. What does this matrix look like if we vectorize it?

$$x = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

Now, write down the corresponding vector *b*

$$\boldsymbol{b} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Using vector *b* and matrix *A* from above, write the augmented matrix.

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Now we solve for vector *x* by using Gaussian Elimination.

1) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

2) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

3) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

4) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

5) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

Set up the new system of equations from your final matrix in your Gaussian Elimination.

What does the $\mathbf{0 = 0}$ tell you about the system of equations?

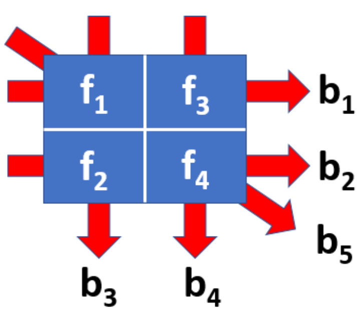
Therefore, according to our rules of Gaussian Elimination, we will end up with __________________ solutions.

Since you are not able to solve for a unique solution, instead solve for $\boldsymbol{f_1, f_2, and\ f_3}$ in terms of $\boldsymbol{f_4}$.

Refer to the "Guess my Square" game from yesterday. What value of f_4 would make your team guess the opposing team's answer?

Making the solution unique

Since the above system gives you infinite solutions, we must now add in more rays. Given the information in the new diagram below, write the new system of equations that corresponds.



Write the matrix-vector equation in the form *Ax=b.*

$$\begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}\begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}$$

If matrix $\boldsymbol{x} = \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$**, write vector *x*.**

$$\boldsymbol{x} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

Now, write down the corresponding vector *b*.

$$\boldsymbol{b} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

Using the new vector *b* and new matrix *A* from above, write the augmented matrix.

$$\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

Now prove that there is a unique solution and solve for vector *x* by using Gaussian Elimination.

1) $\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$

2) $\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$

3) $\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$

4) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

5) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

6) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

7) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

8) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

9) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

Write the new system of equations from the final matrix in your Gaussian Elimination.

Are you able to solve this system of equations? Why?

If yes, solve the system of equations.

☼ **Final Answer:** $\boldsymbol{x} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$ **(Note: You should get the same vector *x* that we have above)**

Lesson 5-Worksheet Student Version

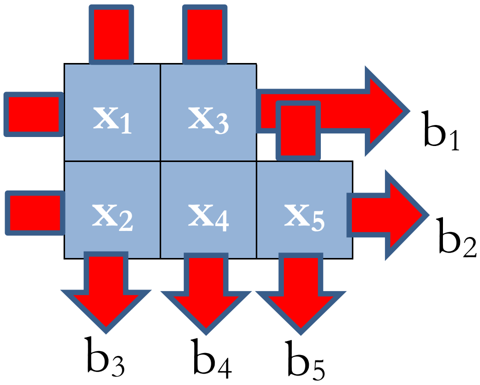
- Student version begins on next page

X-Rays Student Worksheet

Name: ______________________________

Date: __________________ Period: ________

Given the information in the diagram below, write the system of equations that corresponds.



Write the matrix-vector equation in the form *Ax=b*.

$$\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix}\begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

If $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{bmatrix}$**, write down the corresponding vector *b*.**

$$\boldsymbol{b} = \begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

Using vector *b* and matrix *A* from above, write the augmented matrix.

$$\begin{bmatrix} \quad & \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

Now prove that there is a unique solution and solve for vector *x* by using Gaussian Elimination.

1) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

2) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

3) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

4) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

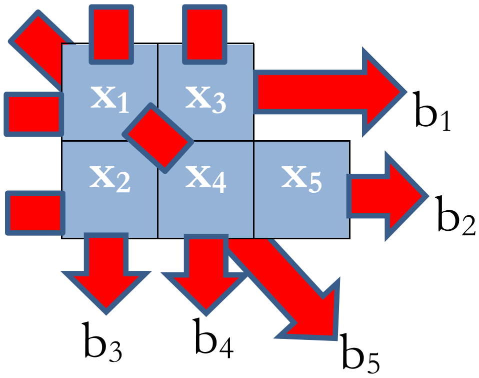
5) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

6) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

7) $\begin{bmatrix} \\ \\ \\ \end{bmatrix}$

As you can see, this will give you infinite solutions. Why did this system give you infinite solutions?

Since the above system gives you infinite solutions, we must now add in more rays. Given the information in the new diagram below, write the new system of equations that corresponds.



Write the matrix-vector equation in the form *Ax=b.*

$$\begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix}\begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

If $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{bmatrix}$**, write down the corresponding vector *b*.**

$$\boldsymbol{b} = \begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

Using the new vector *b* and new matrix *A* from above, write the augmented matrix.

$$\begin{bmatrix} \quad & \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

Now prove that there is a unique solution and solve for vector *x* by using Gaussian Elimination.

1)

2)

3)

4)

5)

6)

7)

8) $\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$

9) $\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$

Write and solve the new system of equations from the final matrix in your Gaussian Elimination.

☼ **Final Answer:** $x = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$ **(Note: You should get the same vector *x* that we have above)**