The Mathematics of Ranking

Authors:
Tamar Avineri, North Carolina School of Science and Mathematics, Durham NC
Emily Berkeley, Panther Creek High School, Cary, NC
Ashley Miller, North Rowan High School, Spencer, NC

Advisors: Eric Hallman, Rachel Minster, Arvind K. Saibaba (asaibab@ncsu.edu)
North Carolina State University

Abstract: In this module, students will engage in activities that apply their knowledge of matrices to the study of ranking. Specifically, students will study Colley's Method of Ranking in various contexts and apply it to a problem that they design in a culminating assignment. The module begins with lessons on elementary matrix operations and then continues with an introduction to the Method through a lesson plan, guided notes, and a problem set. The lesson is followed by a class activity to collect data and apply Colley's Method to that data. The module concludes with a group project in which students work with real-world data to apply their knowledge and communicate their understanding in writing.

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Implementation Notes

- **Length of module:**
  - In total, this unit is designed to take approximately 4.5 days of 90-minute lessons, or 9 days of 45-minute lessons. There is also a final project assessment that, if conducted during class time, is designed to take approximately 120 minutes.
  - Each of the lessons is accompanied by an estimate of the length of time it is designed to take in class. If the estimate is longer than you are able to devote in class, feel free to select portions for students to complete outside of class.

- **Relevant courses:** This module is designed to be self-contained, as the first 3 lessons provide foundational knowledge in the linear algebra skills that students will need for the subsequent lessons. The materials are appropriate for any NC Math 4, Pre-Calculus, or Discrete Mathematics for Computer Science courses. This could also serve as an interesting study following the AP exam for students in AP Calculus AB or BC.

- **Mathematical practices/student learning outcomes:** In addition to the standards for mathematical practices, this module addresses a number of standards covered in NC Math 4, Pre-Calculus and Discrete Mathematics for Computer Science.
  - *Mathematical practices*:
    - Make sense of problems and persevere in solving them.
    - Reason abstractly and quantitatively.
    - Construct viable arguments and critique the reasoning of others.
    - Model with mathematics.
    - Use appropriate tools strategically.
    - Attend to precision.
    - Look for and make use of structure.
    - Look for and express regularity in repeated reasoning.
    - Use strategies and procedures flexibly.
    - Reflect on mistakes and misconceptions.
  - ***NC Math 4:*** NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; NC.M4.N.2.2 Execute procedures of addition, subtraction, and scalar multiplication on vectors.
  - ***Precalculus:*** PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; PC.N.2.5 Execute the multiplication algorithm with matrices.
  - ***Discrete Mathematics for Computer Science*:** DCS.N.1.1 Implement procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; DCS.N.1.2 Implement procedures of addition, subtraction, and scalar multiplication

on vectors; DCS.N.1.3 Implement procedures to find the inverse of a matrix; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation; DCS.N.2.4 Use inverse matrices to solve a system of equations with technology.

- **Lessons 3a and 3b:** There are two lessons on the topic of solving matrix equations; one using inverse matrices and the other using Gaussian elimination. For this module, it is not necessary to cover both lessons. Teachers can choose the lesson that covers their preferred solution approach.

- **Assessments:**
  - Feel free to select portions of the guided notes to serve as out-of-class activities.
  - Any problem set contained within guided notes could be given as homework assignments.
  - As an alternative to the final project in this module, you could choose to give students a standard test or quiz on the skills that have been learned.

- **Online delivery suggestions:**
  - For asynchronous online delivery, create instructional videos to take students through the guided notes.
  - For synchronous online delivery, display the guided notes on your screen and take students through the activities while you annotate on your screen (or writing on paper and using a document camera).
  - Share all prepared documents through a learning management system so that students would have access to them at home

- **Accompanying documents:**
  - Excel Template for Colley's Method Problem Set
  - Colley's Method – Final Project Rubric

- **Student Versions:** Please note that the student versions are located at the end of this document in the Appendix.

Lesson 1: Introduction to Matrices and Matrix Operations

Lesson Plan

Standards NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices.	**Topic/Day: Introduction to Matrices, Matrix Addition/Subtraction, and Scalar Multiplication** **Content Objective: Elementary Matrix Operations** **Vocabulary: matrix; row; column; dimension; square; transpose** **(~60 minutes)**		
	Time	**Student Does**	**Teacher Does**

Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~15 min	Students read the opening problem (e.g., Textbook Problem or other context of interest) from a handout and/or projected on a screen. In groups of 2-3, students talk briefly about how they would answer the question from the teacher. (~5 minutes) The teacher brings back students to share out with the class. (~3 minutes) Students are provided guided notes to document new terms (e.g., matrix, dimension, row, column, etc.) They will complete the notes through the discussion conducted by the teacher. (~5 minutes)	Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, "How might we organize this information in a way that allows us to answer questions about the university's inventory?" After students discuss, the teacher solicits students' responses. If students do not suggest a matrix, the teacher will introduce the name and ask students if they are familiar with the term. If not, the teacher will define it through one of the matrices used to organize the information in the problem.
Explore I Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~15 min	Students complete the second matrix from the problem in their groups. (~5 min) Students work in groups of 2-3 to answer teacher's question. (~5 minutes)	Teacher circulates the room to observe/monitor students' work. Teacher then poses question: "How could we use these matrices to determine the total inventory of books at the university?" Once students have some time to answer question, teacher returns to full class discussion to ask how we could define matrix addition.

Explain I Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~5 min	Students offer their ideas on how to define matrix addition (and subtraction). Students engage in class discussion on teachers' questions.	Teacher conducts discussion on matrix addition and subtraction. Teacher poses questions: "Is matrix addition commutative? Is it associative? Is matrix subtraction commutative? Is it associative? Why/why not?"
Explore II Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~5 min	Students work in groups of 2-3 to answer teacher's question. (~5 minutes)	Teacher then poses question: "How could we use these matrices to determine the inventory of books at the university if the librarian would like to double the inventory?" Once students have some time to answer question, teacher returns to full class discussion to ask how we could define scalar multiplication.
Explain II Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~5 min	Students offer their ideas on how to define scalar multiplication.	Teacher conducts discussion on scalar multiplication.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~15 min	Students complete class problem set in groups of 2-3 to apply their new knowledge.	Teacher circulates the room and observes/monitors students' work.
Evaluate Assessment How will you know if students understand throughout the lesson?	N/A	Students will complete guided notes and a problem set for practice. Students turn in their solutions to the last problem in the problem set as an exit ticket (e.g., Stereo Problem)	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be

			having difficulty in discussion to probe their thinking.

Guided Notes - Teacher Version

Matrix Addition, Subtraction and Scalar Multiplication

A university is taking inventory of the books they carry at their two biggest bookstores.
The East Campus bookstore carries the following books:

Hardcover: Textbooks-5280; Fiction-1680; NonFiction-2320; Reference-1890

Paperback: Textbooks-1930; Fiction-2705; NonFiction-1560; Reference-2130

The West Campus bookstore carries the following books:

Hardcover: Textbooks-7230; Fiction-2450; NonFiction-3100; Reference-1380

Paperback: Textbooks-1740; Fiction-2420; NonFiction-1750; Reference-1170

In order to work with this information, we can represent the inventory of each bookstore using an organized array of numbers known as a *matrix*.

Definitions: A **matrix** is a rectangular table of entries and is used to organize data in a way that can be used to solve problems. The following is a list of terms used to describe matrices:

- A matrix's **size (or dimension)** is written by listing the number of rows "by" the number of columns.
- The values in a matrix, A, are referred to as **entries** or **elements**. The entry in the "m^{th}" row and "n^{th}" column is written as a_{mn}.
- A matrix is **square** if it has the same number of rows as it has columns.
- If a matrix has only one row, then it is a row **vector**. If it has only one column, then the matrix is a column **vector.**

- The **transpose** of a matrix, A, written A^T, switches the rows with the columns of A and the columns with the rows.
- Two matrices are **equal** if they have the same size and the same corresponding entries.

The inventory of the books at the East Campus bookstore can be represented with the following **2 x 4** matrix:

$$E = \begin{array}{c} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} \end{array}$$

Similarly, the West Campus bookstore's inventory can be represented with the following matrix:

$$W = \begin{array}{c} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 7230 & 2450 & 3100 & 1380 \\ 1740 & 2420 & 1750 & 1170 \end{bmatrix} \end{array}$$

Adding and Subtracting Matrices

In order to add or subtract matrices, they must first be of the same **size**. The result of the addition or subtraction is a matrix of the same size as the matrices themselves, and the entries are obtained by adding or subtracting the elements in corresponding positions.

In our campus bookstores example, we can find the total inventory between the two bookstores as follows:

$$E + W = \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} + \begin{bmatrix} 7230 & 2450 & 3100 & 1380 \\ 1740 & 2420 & 1750 & 1170 \end{bmatrix}$$

$$= \begin{array}{c} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 12510 & 4130 & 5420 & 3270 \\ 3670 & 5125 & 3310 & 3300 \end{bmatrix} \end{array}$$

Question: *Is matrix addition commutative* $(e.g., A + B = B + A)$*? Why or why not?*
Matrix addition is commutative. This is because the operation is based in the addition of real numbers, as the entries of each matrix are added to their corresponding entries in the other matrix/matrices. Since addition of real numbers is commutative, so is matrix addition.

Question: *Is matrix subtraction commutative* $(e.g., A - B = B - A)$*? Why or why not?*
Matrix subtraction is not commutative. This is because the operation is based in the subtraction of real numbers, as the entries of each matrix are subtracted from their corresponding entries in the other matrix/matrices. Since subtraction of real numbers is not commutative, neither is matrix subtraction.

Question: *Is matrix addition associative* $(e.g., (A + B) + C = A + (B + C))$*? Why or why not?*
Matrix addition is associative. This is because the operation is based in the addition of real numbers, as the entries of each matrix are added to their corresponding entries in the other matrix/matrices. Since addition of real numbers is associative, so is matrix addition.

Question: *Is matrix subtraction associative* $(e.g., (A - B) - C = A - (B - C))$*? Why or why not?*
Matrix subtraction is not associative. This is because the operation is based in the subtraction of real numbers, as the entries of each matrix are subtracted from their corresponding entries in the other matrix/matrices. Since subtraction of real numbers is not associative, neither is matrix subtraction.

Scalar Multiplication

Multiplying a matrix by a constant (or *scalar*) is as simple as multiplying each entry by that number! Suppose the bookstore manager in East Campus wants to double his inventory. He can find the number of books of each type that he would need by simply multiplying the matrix *E* by the scalar (or constant) 2. The result is as follows:

$$2E = 2 * \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 2(5280) & 2(1680) & 2(2320) & 2(1890) \\ 2(1930) & 2(2705) & 2(1560) & 2(2130) \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{r} Hardback \\ Paperback \end{array} & \begin{bmatrix} 10560 & 3360 & 4640 & 3780 \\ 3860 & 5410 & 3120 & 4260 \end{bmatrix} \end{array}$$

Exercises: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed:

a. $A + B$: This operation cannot be performed, since matrices A and B are of different dimensions.

b. $B - A$: This operation also cannot be performed, as A and B have different dimensions.

c. $A - C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-6} & \mathbf{22} \\ \mathbf{0} & \mathbf{-8} & \mathbf{12} \\ \mathbf{-11} & \mathbf{8} & \mathbf{7} \end{bmatrix}$

d. $C - A = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{6} & \mathbf{-22} \\ \mathbf{0} & \mathbf{8} & \mathbf{-12} \\ \mathbf{11} & \mathbf{-8} & \mathbf{-7} \end{bmatrix}$

e. $5B = 5 * \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{40} & \mathbf{-30} \end{bmatrix}$

f. $-A + 4C = -\begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} + 4 * \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} =$

$$\begin{bmatrix} -1 & 0 & -1 \\ -2 & 4 & -3 \\ 6 & -1 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 24 & -84 \\ 8 & 16 & -36 \\ 20 & -28 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{24} & \mathbf{-85} \\ \mathbf{6} & \mathbf{20} & \mathbf{-39} \\ \mathbf{26} & \mathbf{-29} & \mathbf{-4} \end{bmatrix}$$

g. $B - D$: This operation cannot be performed, since B and D are not of the same size.

h. $2C - 6A = 2 * \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} - 6 * \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} =$

$$\begin{bmatrix} 0 & 12 & -42 \\ 4 & 8 & -18 \\ 10 & -14 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 6 \\ 12 & -24 & 18 \\ -36 & 6 & 48 \end{bmatrix} = \begin{bmatrix} \mathbf{-6} & \mathbf{12} & \mathbf{-48} \\ \mathbf{-8} & \mathbf{32} & \mathbf{-36} \\ \mathbf{46} & \mathbf{-20} & \mathbf{-46} \end{bmatrix}$$

i. $B^T + D = \begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{7} \\ \mathbf{6} \\ \mathbf{-3} \end{bmatrix}$

Lesson 2: Matrix Multiplication

Lesson Plan

Standards	**Topic/Day: Matrix Multiplication** **Content Objective: Elementary Matrix Operations**
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context	**(~70 minutes)**

	Time	**Student Does**	**Teacher Does**
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~5 min	Students read the opening problem (e.g., Opera Problem or other context of interest) from a handout and/or projected on a screen.	Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, "How might we organize this information in a way that allows us to answer the question?"

Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~15 min	In groups of 2-3, students work together to calculate each value of interest by hand (not using any specific method). (~10 minutes) Students work in groups of 2-3 to answer teacher's question. (~10 minutes) Students can break the work up among their group members.	Teacher asks students to calculate each value of interest by hand, showing their work but not using any specific method. The teacher brings students back to share their results and confirm their results with other groups.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~15 min	Students follow along the teachers' explanation on their opening problem. Students share their thoughts on teacher's posed questions.	Teacher conducts lesson on matrix multiplication using the opening problem to demonstrate the operation. Teacher poses questions: "Is matrix multiplication commutative? Is it associative? Why/why not?" Teacher provides examples of why they are/aren't, and students practice the operation with those examples.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~25 min	Students complete class problem set in groups of 2-3 to apply their new knowledge.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate Assessment How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.

Guided Notes - Teacher Version

Matrix Multiplication

The Metropolitan Opera is planning its last cross-country tour. It plans to perform *Carmen* and *La Traviata* in Atlanta in May. The person in charge of logistics wants to make plane reservations for the two troupes. *Carmen* has 2 stars, 25 other adults, 5 children, and 5 staff members. *La Traviata* has 3 stars, 15 other adults, and 4 staff members. There are 3 airlines to choose from. Redwing charges round-trip fares to Atlanta of $630 for first class, $420 for coach, and $250 for youth. Southeastern charges $650 for first class, $350 for coach, and $275 for youth. Air Atlanta charges $700 for first class, $370 for coach, and $150 for youth. Assume stars travel first class, other adults and staff travel coach, and children travel for the youth fare.

Use multiplication and addition to find the total cost for each troupe to travel each of the airlines.

Carmen/Redwing: $2(630) + 30(420) + 5(250) = \$15110$

Carmen/Southeastern: $2(650) + 30(350) + 5(275) = \$13175$

Carmen/Air Atlanta: $2(700) + 30(370) + 5(150) = \$13250$

La Traviata/Redwing: $3(630) + 19(420) + 0(250) = \$9870$

La Traviata/Southeastern: $3(650) + 19(350) + 0(275) = \$8600$

La Traviata/Air Atlanta: $3(700) + 19(370) + 0(150) = \$9130$

It turns out that we can solve problems like these using a matrix operation, specifically **matrix multiplication**!

We first note that matrix multiplication is only defined for matrices of certain sizes. For the product AB of matrices A and B, where A is an $m\ x\ n$ matrix, B must have the same number of rows as A has columns. So, B must have size $\boldsymbol{n}\ x\ p$. The product AB will have size $\boldsymbol{m\ x\ p}$.

Exercises

The following is a set of abstract matrices (without row and column labels):

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad O = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 5 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

List at least 5 orders of pairs of matrices from this set for which the product is defined. State the dimension of each product.

MO: 2x1 **MP: 2x2** **PM: 2x2** **MR: 2x2** **RM: 2x2** **NQ: 3x1**

NU: 3x4 **PO: 2x1** **US: 3x2** **UT: 3x1**

Back to the opera…

Define two matrices that organize the information given:

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{stars} & \textbf{adults} & \textbf{children} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 2 & 30 & 5 \\ 3 & 19 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} & \begin{bmatrix} 630 & 650 & 700 \\ 420 & 350 & 370 \\ 250 & 275 & 150 \end{bmatrix} \end{array}$$

We can multiply these two matrices to obtain the same answers we obtained above, all in one matrix!

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{stars} & \textbf{adults} & \textbf{children} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 2 & 30 & 5 \\ 3 & 19 & 0 \end{bmatrix} \end{array} \cdot \begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} & \begin{bmatrix} 630 & 650 & 700 \\ 420 & 350 & 370 \\ 250 & 275 & 150 \end{bmatrix} \end{array}$$

$$= \begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 15110 & 13175 & 13250 \\ 9870 & 8600 & 9130 \end{bmatrix} \end{array}$$

Carmen/Redwing: $15110

Carmen/Southeastern: $13175

Carmen/Air Atlanta: $13250

La Traviata/Redwing: $9870

La Traviata/Southeastern: $8600

La Traviata/Air Atlanta: $9130

Exercises

1. The K.L. Mutton Company has investments in three states - North Carolina, North Dakota, and New Mexico. Its deposits in each state are divided among bonds, mortgages, and consumer loans. The amount of money (in millions of dollars) invested in each category on June 1 is displayed in the table below.

	NC	ND	NM
Bonds	13	25	22
Mort.	6	9	4
Loans	29	17	13

The current yields on these investments are 7.5% for bonds, 11.25% for mortgages, and 6% for consumer loans. Use matrix multiplication to find the total earnings for each state.

Total earnings for each state (in millions of dollars):

$$\begin{array}{ccc} \textbf{Bonds} & \textbf{Mort.} & \textbf{Loans} \end{array} \atop \begin{bmatrix} 1.075 & 1.1125 & 1.06 \end{bmatrix} \quad \begin{array}{c|ccc} & NC & ND & NM \\ \hline \textbf{Bonds} & 13 & 25 & 22 \\ \textbf{Mort.} & 6 & 9 & 4 \\ \textbf{Loans} & 29 & 17 & 13 \end{array} = \begin{array}{ccc} NC & ND & NM \\ \hline 3.39 & 3.9075 & 2.88 \end{array}$$

2. Several years ago, Ms. Allen invested in growth stocks, which she hoped would increase in value over time. She bought 100 shares of stock A, 200 shares of stock B, and 150 shares of stock C. At the end of each year she records the value of each stock. The table below shows the price per share (in dollars) of stocks A, B, and C at the end of the years 1984, 1985, and 1986.

	1984	1985	1986
Stock A	68.00	72.00	75.00
Stock B	55.00	60.00	67.50
Stock C	82.50	84.00	87.00

Calculate the total value of Ms. Allen's stocks at the end of each year.

Total value of the stocks (in dollars) at the end of each year:

$$\begin{array}{ccc} A & B & C \end{array} \atop \begin{bmatrix} 100 & 200 & 150 \end{bmatrix} \quad \begin{array}{c|ccc} & 1984 & 1985 & 1986 \\ \hline A & 68 & 72 & 75 \\ B & 55 & 60 & 67.5 \\ C & 82.5 & 84 & 87 \end{array} = \begin{array}{ccc} 1984 & 1985 & 1986 \\ \hline 30,175 & 31,800 & 34,050 \end{array}$$

3. The Sound Company produces stereos. Their inventory includes four models - the Budget, the Economy, the Executive, and the President models. The Budget needs 50 transistors, 30 capacitors, 7 connectors, and 3 dials. The Economy model needs 65 transistors, 50 capacitors, 9 connectors, and 4 dials. The Executive model needs 85 transistors, 42 capacitors, 10 connectors, and 6 dials. The President model needs 85 transistors, 42 capacitors, 10 connectors, and 12 dials. The daily manufacturing goal in a normal quarter is 10 Budget, 12 Economy, 11 Executive, and 7 President stereos.

   a. How many transistors are needed each day? Capacitors? Connectors? Dials?

   b. During August and September, production is increased by 40%. How many Budget, Economy, Executive, and President models are produced daily during these months?

   c. It takes 5 person-hours to produce the Budget model, 7 person-hours to produce the Economy model, 6 person-hours for the Executive model, and 7 person-hours for the President model. Determine the number of employees needed to maintain the normal production schedule, assuming everyone works an average of 7 hours each day. How many employees are needed in August and September?

Define the matrices for the inventory parts ($\boldsymbol{I}$) and the daily manufacturing goal ($\boldsymbol{N}$) as

$$\boldsymbol{I} = \begin{array}{c c} & \begin{array}{cccc} t & ca & co & d \end{array} \\ \begin{array}{c} B \\ Ec \\ Ex \\ P \end{array} & \begin{bmatrix} 50 & 30 & 7 & 3 \\ 65 & 50 & 9 & 4 \\ 85 & 42 & 10 & 6 \\ 85 & 42 & 10 & 12 \end{bmatrix} \end{array} \quad and \quad \boldsymbol{N} = \begin{array}{c} \begin{array}{cccc} B & Ec & Ex & P \end{array} \\ \begin{bmatrix} 10 & 12 & 11 & 7 \end{bmatrix} \end{array}$$

a. The answers are the results of the matrix multiplication

$$\boldsymbol{NI} = \begin{array}{c} \begin{array}{cccc} t & ca & co & d \end{array} \\ \begin{bmatrix} 2810 & 1656 & 358 & 228 \end{bmatrix} \end{array}$$

b. The new daily manufacturing goals are given by

$$\boldsymbol{1.4N} = \begin{array}{c} \begin{array}{cccc} B & Ec & Ex & P \end{array} \\ \begin{bmatrix} 14 & 16.8 & 15.4 & 9.8 \end{bmatrix} \end{array}$$

Which should be rounded to integer quantities

c. Define a matrix $\boldsymbol{H}$ for hours of labor as

$$\boldsymbol{H} = \begin{array}{c c} & Hrs. \\ \begin{array}{c} B \\ Ec \\ Ex \\ P \end{array} & \begin{bmatrix} 5 \\ 7 \\ 6 \\ 7 \end{bmatrix} \end{array}$$

The number of labor hours needed per week is given by

$$\boldsymbol{NH = 249}$$

With 7-hour workdays, the number of employees needed is $\frac{249}{7} = 35.6$, which implies that 36 employees are needed to maintain full production. For August and September, we want $\frac{1.4NH}{7} = \frac{348.6}{7}$, which rounds to 50.

4. The president of the Lucrative Bank is hoping for a 21% increase in checking accounts, a 35% increase in savings accounts, and a 52% increase in market accounts. The current statistics on the number of accounts at each branch are as follows:

$$\begin{array}{r c} & \begin{array}{ccc} \textit{Checking} & \textit{Savings} & \textit{Market} \end{array} \\ \begin{array}{r} \textit{Northgate} \\ \textit{Downtown} \\ \textit{South Square} \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array}$$

What is the goal for each branch in each type of account? (HINT: multiply by a 3×2 matrix with certain nonzero entries on the diagonal and zero entries elsewhere.) What will be the total number of accounts at each branch?

The goal for each branch in each type of account is given by:

$$\begin{array}{r c} & \begin{array}{ccc} \boldsymbol{c} & \boldsymbol{s} & \boldsymbol{m} \end{array} \\ \begin{array}{r} \boldsymbol{N} \\ \boldsymbol{S} \\ \boldsymbol{D} \end{array} & \begin{bmatrix} \mathbf{40039} & \mathbf{10135} & \mathbf{512} \\ \mathbf{15231} & \mathbf{8751} & \mathbf{105} \\ \mathbf{25612} & \mathbf{12187} & \mathbf{97} \end{bmatrix} \end{array} \cdot \begin{array}{r c} & \begin{array}{ccc} \boldsymbol{c} & \boldsymbol{s} & \boldsymbol{m} \end{array} \\ \begin{array}{r} \boldsymbol{c} \\ \boldsymbol{s} \\ \boldsymbol{m} \end{array} & \begin{bmatrix} \mathbf{1.21} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1.35} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1.52} \end{bmatrix} \end{array}$$

$$= \begin{array}{r c} & \begin{array}{ccc} \boldsymbol{c} & \boldsymbol{s} & \boldsymbol{m} \end{array} \\ \begin{array}{r} \boldsymbol{N} \\ \boldsymbol{S} \\ \boldsymbol{D} \end{array} & \begin{bmatrix} \mathbf{48447} & \mathbf{13682} & \mathbf{778.24} \\ \mathbf{18430} & \mathbf{11814} & \mathbf{159.6} \\ \mathbf{30991} & \mathbf{16452} & \mathbf{147.44} \end{bmatrix} \end{array}$$

Right-multiplying this result by the matrix $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$ yields the following total number of accounts at each branch: $\begin{array}{r c} & \boldsymbol{Total} \\ \begin{array}{r} \boldsymbol{N} \\ \boldsymbol{D} \\ \boldsymbol{S} \end{array} & \begin{bmatrix} \mathbf{62907.68} \\ \mathbf{30402.96} \\ \mathbf{47590.41} \end{bmatrix} \end{array}.$

Note: this answer can also be obtained by just adding up the entries in each row of the previous matrix.

Lesson 3a: Solving Linear Systems of Equations Using Inverse Matrices

Lesson Plan

<table>
<tr><td colspan="2">Standards

DCS.N.1.3 Implement procedures to find the inverse of a matrix; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation; DCS.N.2.4 Use inverse matrices to solve a system of equations with technology.</td><td colspan="2">Topic/Day: Solving Linear Systems of Equations Using the Inverse of a Matrix
Vocabulary: (multiplicative) identity matrix; (multiplicative) inverse matrix

(~75 minutes)</td></tr>
<tr><td></td><td>Time</td><td>Student Does</td><td>Teacher Does</td></tr>
<tr><td>Warm Up
Elicit/Engage
Build relevance through a problem
Try to find out what your students already know
Get them interested</td><td>~5 min</td><td>Students read the opening problem (e.g., Business Problem or other context of interest) from a handout and/or projected on a screen.

Students work in groups of 2-3 to represent the problem with a system of equations.</td><td>Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, “How might we represent this problem with a system of equations?”</td></tr>
<tr><td>Explore
Connectivity to build understanding of concepts
Allow for collaboration consider heterogeneous groups
Move deliberately from concrete to abstract
Apply scaffolding & personalization</td><td>~10 min</td><td>Students work in groups of 2-3 to answer teacher’s question.</td><td>Teacher asks students to consider how they could use matrices to represent the system of equations as a matrix equation.

The teacher brings students back to share their results.</td></tr>
</table>

Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~30 min	Students follow along the teachers' explanation on a problem out of context. Students work together on practice problems based on the teacher's lesson.	Teacher conducts lesson on solving a matrix equation using a non-contextual problem. Teacher introduces the concept of the inverse of a matrix during this part of the lesson. Teacher includes a tutorial on using the calculator to calculate the inverse of a matrix.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~20 min	Students apply their new understanding to the opening problem.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate **Assessment** How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.

Guided Notes - Teacher Version

Solving Linear Systems of Equations Using Inverse Matrices

A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling.

How can we represent this problem with a system of equations?

Let x = amount of money for the East location

Let y = amount of money for the West location

Let z = amount of money for the South location

We therefore have the following system of equations:

$$\begin{aligned} 0.5x + 0.3y + 0.4z &= 43{,}000 \\ 0.2x + 0.3y + 0.4z &= 28{,}000 \\ 0.3x + 0.4y + 0.2z &= 29{,}000 \end{aligned}$$

Definitions:

- The **multiplicative identity** of a square $n\ x\ n$ matrix, A, is an $n\ x\ n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$.
- If an $n\ x\ n$ matrix A^{-1} exists such that $AA^{-1} = I$, then A^{-1} is the **multiplicative inverse** of A. (Note that not all matrices have **inverses**. For example, no rectangular matrix (e.g., $2\ x\ 3$) has an **inverse**.)

Example: Consider the following system of linear equations (recall this from Algebra II):

$$\begin{gathered} x + 3y = 0 \\ x + y + z = 1 \\ 3x - y - z = 11 \end{gathered}$$

We can solve this system by representing it using matrices.

We will name the **coefficient** matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$, the **variable vector** $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the **column vector** $B = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$. So, our **matrix equation** (also referred to as a linear system of equations) representing the system can be written as $AX = B$:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

Note: Division is not an operation that is defined for matrices. The analogous operation, however, is multiplying by the **inverse** of a matrix. Just as we divide in order to "reverse" the operation of multiplication between real numbers to return the number 1 (the multiplicative identity in real numbers), we multiply matrices by their inverses to "reverse" the operation of multiplication between matrices, returning the identity matrix, I.

So, in order to solve the equation $AX = B$ for the matrix X, we will need to do the following, as long as A^{-1} exists:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

So, back to our problem:

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

We use out calculator to find the inverse of the coefficient matrix, which is

$$\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & -\frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & -\frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

The solution to our system, then, is $x = \mathbf{3}$, $y = \mathbf{-1}$ and $z = \mathbf{-1}$.

Recall: A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling. How much money will each location receive in grants?

Rewrite your system of equations from earlier in this lesson:

$$\begin{aligned} 0.5x + 0.3y + 0.4z &= 43{,}000 \\ 0.2x + 0.3y + 0.4z &= 28{,}000 \\ 0.3x + 0.4y + 0.2z &= 29{,}000 \end{aligned}$$

We can represent this system using the following linear system of equations:

$$\begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 43000 \\ 28000 \\ 29000 \end{bmatrix}$$

Using our calculators to find the inverse of the coefficient matrix $A = \begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}$ we have $A^{-1} \approx \begin{bmatrix} 3.333 & -3.333 & 0 \\ -2.667 & 0.667 & 4 \\ 0.333 & 3.667 & -3 \end{bmatrix}$. Since the equation $AX = B$ can be solved by $X = A^1B$, we find

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} 3.333 & -3.333 & 0 \\ -2.667 & 0.667 & 4 \\ 0.333 & 3.667 & -3 \end{bmatrix} \begin{bmatrix} 43000 \\ 28000 \\ 29000 \end{bmatrix} = \begin{bmatrix} 50{,}000 \\ 20{,}000 \\ 30{,}000 \end{bmatrix}$$

Therefore, **$50,000** goes to the East location, **$20,000** goes to the West location, and **$30,000** goes to the South location.

Exercises

For each of the following problems, identify your variables and write a system of equations to represent the problem. Then use matrices to solve the system.

1. The Frodo Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat is $42 and $30 per acre, respectively. Mr. Frodo has $18,600 available for cultivating these crops. If he wants to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant? (Adapted from *Finite Mathematics,* Tan p. 93 #51[1])

   Let x = number of acres of corn
   y = number of acres of wheat

$$42x + 30y = 18600$$
$$x + y = 500$$

$$\begin{bmatrix} 42 & 30 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18600 \\ 500 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 300 \\ 200 \end{bmatrix}$$

$$x = 300\,, \qquad y = 200$$

300 acres of corn and 200 acres of wheat should be cultivated.

2. The Coffee Cart sells a blend made with two different coffees, one costing $2.50 per pound, and the other costing $3.00 per pound. If the blended coffee sells for $2.80 per pound, how much of each coffee is used to obtain the blend? (Assume that the weight of the coffee blend is 100 pounds.) (Adapted from *Finite Mathematics,* Tan p. 93 #53)

   Let x = number of pounds of $2.50 coffee
   y = number of pounds of $3.00 coffee

$$2.50x + 3.00y = 280$$
$$x + y = 100$$

$$\begin{bmatrix} 2.50 & 3.00 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 280 \\ 100 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

$$x = 40\,, \quad y = 60$$

40 lbs of Coffee 1 should be blended with 60 lbs of Coffee 2 to make the proper blend.

[1] Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

3. The Maple Movie Theater has a seating capacity of 900 and charges \$2 for children, \$3 for students, and \$4 for adults. At a screening with full attendance last week, there were half as many adults as children and students combined. The receipts totaled \$2800. How many adults attended the show? (Adapted from *Finite Mathematics,* Tan p. 97 #60)

Let x = number of children who attended the show
y = number of students who attended the show
z = number of adults who attended the show

$$\begin{gathered} x+y+z=2800 \\ 2x+3y+4z=900 \\ x+y-2z=0 \end{gathered}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2,800 \\ 900 \\ 0 \end{bmatrix}$$

$$A^{-1}B=\begin{bmatrix} 200 \\ 400 \\ 300 \end{bmatrix} \quad \rightarrow \quad x=200\,,\ \ y=400\,,\ \ z=300$$

200 Children, 400 Students, and 300 adults attended.

4. The Toolies have a total of \$100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12% per year, while bonds pay 8% per year, and the money market account pays 4% per year. They have decided that the amount invested in stocks should be equal to the difference between the amount invested in bonds and 3 times the amount invested in the money market account. How should the Toolies allocate their resources if they require an annual income of \$10,000 from their investments? (Adapted from *Finite Mathematics,* Tan p. 106 #36)

Let x = amount allocated to stocks
y = amount allocated to bonds
z = amount allocated to a money market account

$$\begin{gathered} x+y+z=100{,}000 \\ .12x+.08y+.04z=10{,}000 \\ x-y+3z=0 \end{gathered}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ .12 & .08 & .04 \\ 1 & -1 & 3 \end{bmatrix}\times\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 100{,}000 \\ 10{,}000 \\ 0 \end{bmatrix}$$

$$A^{-1}B=\begin{bmatrix} 50{,}000 \\ 50{,}000 \\ 0 \end{bmatrix} \quad \rightarrow \quad x=50{,}000\,,\ \ y=50{,}000\,,\ \ z=0$$

\$50,000 should be put into the stock market, \$50,000 in bonds, and no investment should be made in a Money Market Account.

Lesson 3b: Solving Linear Systems of Equations Using Gaussian Elimination

Lesson Plan

Standards DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation	**Topic/Day: Solving Linear Systems of Equations Using Gaussian Elimination** **Vocabulary: Gaussian elimination; row reduction** **(~75 minutes)**		
	Time	**Student Does**	**Teacher Does**
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~5 min	Students read the opening problem (e.g., Business Problem or other context of interest) from a handout and/or projected on a screen. Students work in groups of 2-3 to represent the problem with a system of equations.	Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, "How might we represent this problem with a system of equations?
Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~10 min	Students work in groups of 2-3 to answer teacher's question.	Teacher asks students to consider how they could use matrices to represent the system of equations as a matrix equation. The teacher brings students back to share their results.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~30 min	Students follow along the teachers' explanation on a problem out of context. Students work together on practice problems based on the teacher's lesson.	Teacher conducts lesson on solving a matrix equation using a non-contextual problem. Teacher introduces the method of Gaussian

			Elimination during this part of the lesson. Teacher includes a tutorial on using the calculator to apply Gaussian Elimination.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~20 min	Students apply their new understanding to the opening problem.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate **Assessment** How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.

Guided Notes - Teacher Version

Solving Linear Systems of Equations Using Gaussian Elimination

A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling.

How can we represent this problem with a system of equations?

Let x = amount of money for the East location
Let y = amount of money for the West location
Let z = amount of money for the South location

We therefore have the following system of equations:

$$\begin{aligned} 0.5x + 0.3y + 0.4z &= 43{,}000 \\ 0.2x + 0.3y + 0.4z &= 28{,}000 \\ 0.3x + 0.4y + 0.2z &= 29{,}000 \end{aligned}$$

Example: Consider the following system of linear equations (recall this from Algebra II):

$$\begin{aligned} x + 3y &= 0 \\ x + y + z &= 1 \\ 3x - y - z &= 11 \end{aligned}$$

We can solve this system by representing it using matrices.

We will name the **coefficient** matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$, the **variable vector** $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the **column vector** $B = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$. So, our **matrix equation** (also referred to as a linear system of equations) representing the system can be written as $AX = B$:

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

One way to solve this system is to use an approach known as **Gaussian elimination**, or **row reduction.**

Gaussian Elimination

You may recall from your prior mathematics work that there are three possible conclusions we can make about the solution to a system of equations.

Case 1: There exists one unique solution.
Case 2: There is no solution.
Case 3: There is an infinite number of solutions.

Case 1: There exists one unique solution.

Recall our example from above:

$$\begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

To begin, we write the associated **augmented matrix**, which is written in the following form:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$$

To apply the method on a matrix, we use **elementary row operations** to modify the matrix. Our goal is to end up with the **identity matrix**, which is an $n \ x \ n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, on the left side of the augmented matrix.

Our solution to the system of equations will be the resulting matrix on the right side of the augmented matrix. This is because the resulting augmented matrix would represent a system of equations in which each variable could be solved for (if a solution exists).

Elementary Row Operations:

There are three operations that can be applied to modify the matrix and still preserve the solution to the system of equations.

- Exchanging two rows (which represents the switching the listing order of two equations in the system)

- Multiplying a row by a nonzero scalar (which represents multiplying both sides of one of the equations by a nonzero scalar)
- Adding a multiple of one row to another (which represents does not affect the solution, since both equations are in the system)

For our example…

$$\begin{array}{ll} x+3y=0 & R_1 \\ x+y+z=1 & R_2 \\ 3x-y-z=11 & R_3 \end{array}$$

System of equations	Row operation	Augmented matrix
$x+3y=0$ $x+y+z=1$ $3x-y-z=11$		$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$
$x+3y=0$ $-y+z=1$ $3x-y-z=11$	$R_2-R_1\to R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 3 & -1 & -1 & 11 \end{array}\right]$
$x+3y=0$ $-y+z=1$ $-10y-z=11$	$R_3-3R_1\to R_3$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x+3y=0$ $-12y=12$ $-10y-z=11$	$R_2+R_3\to R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & -12 & 0 & 12 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x+3y=0$ $y=-1$ $-10y-z=11$	$-\frac{1}{12}R_2\to R_2$	$\left[\begin{array}{ccc\|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x=3$ $y=-1$ $-10y-z=11$	$R_1-3R_2\to R_1$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -10 & -1 & 11 \end{array}\right]$
$x=3$ $y=-1$ $-z=1$	$R_3+10R_2\to R_3$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$x=3$ $y=-1$ $z=-1$	$-R_3\to R_3$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

The solution to our system is therefore $\boldsymbol{x=3, y=-1}$ and $\boldsymbol{z=-1}$.

Back to our opening problem! A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling. How much money will each location receive in grants?

Rewrite your system of equations from earlier in this lesson:

$$\begin{gathered} 0.5x + 0.3y + 0.4z = 43{,}000 \\ 0.2x + 0.3y + 0.4z = 28{,}000 \\ 0.3x + 0.4y + 0.2z = 29{,}000 \end{gathered}$$

We can represent this system using the following systems of linear equations:

$$\begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.3 & 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 43000 \\ 28000 \\ 29000 \end{bmatrix}$$

The augmented matrix for this system is:

$$\left[\begin{array}{ccc|c} 0.5 & 0.3 & 0.4 & 43000 \\ 0.2 & 0.3 & 0.4 & 28000 \\ 0.3 & 0.4 & 0.2 & 29000 \end{array}\right]$$

Using elementary row operations, we find that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} 50{,}000 \\ 20{,}000 \\ 30{,}000 \end{bmatrix}$$

So, **$50,000** goes to the East location, **$20,000** goes to the West location, and **$30,000** goes to the South location.

<u>Case 2</u>: There is no solution.

Consider the system of equations:

$$\begin{aligned} &2x - y + z = 1 \\ &3x + 2y - 4z = 4 \\ &-6x + 3y - 3z = 2 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{-6} & \mathbf{3} & \mathbf{-3} & \mathbf{2} \end{array}\right]$

Using row operation $R_3 + 3R_1 \to R_3$, we get $\left[\begin{array}{ccc|c} \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{2} & \mathbf{-4} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5} \end{array}\right]$.

We note that the third row in the augmented matrix is a false statement, so there is no solution to this system.

<u>Case 3</u>: There is an infinite number of solutions.

Consider the system of equations:

$$\begin{aligned} &x - y + 2z = -3 \\ &4x + 4y - 2z = 1 \\ &-2x + 2y - 4z = 6 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{4} & \mathbf{4} & \mathbf{-2} & \mathbf{1} \\ \mathbf{-2} & \mathbf{2} & \mathbf{-4} & \mathbf{6} \end{array}\right]$

Using row operations $R_2 - 4R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$, we get $\left[\begin{array}{ccc|c} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{8} & \mathbf{-10} & \mathbf{13} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$.

This represents a system that leaves us with 2 equations and 3 unknowns. So, we are unable to solve for one variable without expressing it in terms of another. This gives us an infinite number of solutions.

Exercises

For each of the following problems, identify your variables and write a system of equations to represent the problem. Then use matrices to solve the system.

1. The Frodo Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat is \$42 and \$30 per acre, respectively. Mr. Frodo has \$18,600 available for

cultivating these crops. If he wants to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant? (Adapted from *Finite Mathematics,* Tan p. 93 #51[2])

Let x = number of acres of corn
y = number of acres of wheat

$$\begin{aligned} 42x + 30y &= 18600 \\ x + y &= 500 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{cc|c} 42 & 30 & 18600 \\ 1 & 1 & 500 \end{array}\right]$

Solution: $x = 300$, $y = 200$

300 acres of corn and 200 acres of wheat should be cultivated.

2. The Coffee Cart sells a blend made with two different coffees, one costing \$2.50 per pound, and the other costing \$3.00 per pound. If the blended coffee sells for \$2.80 per pound, how much of each coffee is used to obtain the blend? (Assume that the weight of the coffee blend is 100 pounds.) (Adapted from *Finite Mathematics,* Tan p. 93 #53)

Let x = number of pounds of \$2.50 coffee
y = number of pounds of \$3.00 coffee

$$\begin{aligned} 2.50x + 3.00y &= 280 \\ x + y &= 100 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{cc|c} 2.5 & 3 & 280 \\ 1 & 1 & 100 \end{array}\right]$

Solution: $x = 40$, $y = 60$

40 lbs of Coffee 1 should be blended with 60 lbs of Coffee 2 to make the proper blend.

3. The Maple Movie Theater has a seating capacity of 900 and charges \$2 for children, \$3 for students, and \$4 for adults. At a screening with full attendance last week, there were half as many adults as children and students combined. The receipts totaled \$2800. How many adults attended the show? (Adapted from *Finite Mathematics,* Tan p. 97 #60)

Let x = number of children who attended the show
y = number of students who attended the show
z = number of adults who attended the show

[2] Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

$$\begin{aligned} x+y+z &= 2800 \\ 2x+3y+4z &= 900 \\ x+y-2z &= 0 \end{aligned}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2800 \\ 2 & 3 & 4 & 900 \\ 1 & 1 & -2 & 0 \end{array}\right]$

Solution: $x = 200$, $y = 400$, $z = 300$

200 children, 400 students, and 300 adults attended.

4. The Toolies have a total of $100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12% per year, while bonds pay 8% per year, and the money market account pays 4% per year. They have decided that the amount invested in stocks should be equal to the difference between the amount invested in bonds and 3 times the amount invested in the money market account. How should the Toolies allocate their resources if they require an annual income of $10,000 from their investments? (Adapted from *Finite Mathematics,* Tan p. 106 #36)

Let x = amount allocated to stocks
y = amount allocated to bonds
z = amount allocated to a money market account

$$\begin{aligned} x+y+z &= 100{,}000 \\ .12x+.08y+.04z &= 10{,}000 \\ x-y+3z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ .12 & .08 & .04 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 100{,}000 \\ 10{,}000 \\ 0 \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100{,}000 \\ .12 & .08 & .04 & 10{,}000 \\ 1 & -1 & 3 & 0 \end{array}\right]$

Solution: $x = 50{,}000$, $y = 50{,}000$, $z = 0$

$50,000 should be put into the stock market, $50,000 in bonds, and no investment should be made in a Money Market Account.

Lesson 4: Introduction to Colley's Method

Lesson Plan

Standards		**Topic/Day:** Intro to Colley's Method **Content Objective:** Students will be able to understand how Colley's method is used to rank groupings **Vocabulary:** Colley's Method, Ranking **Materials Needed:** Video playing platform (TV or projector), Guided notes (printed if needed), Method to collect data, Method to create teams, Exit ticket/quiz (prepared and a method of delivery – printed or Google Forms) **(~85 minutes)**	
DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation			
	Time	**Student Does**	**Teacher Does**
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~10 mins	Watch video introducing rankings Discussion about the video and ranking in general.	Show video Help students navigate discussion – you can ask the questions outlined, or you can turn them into a handout for the students
Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~35 mins	Complete Lesson 4 - Guided Notes on Intro to Colley's Method. Students will follow along with the teacher through the vocabulary, definitions, and first example. They will then work on Example 2 independently or in small groups.	Work through the guided notes with the students. You will complete up through Example 1 with the students. Help get the students started on filling in the table for Example 2 before sending them off to complete this on their own or in small groups. Note: You will want to determine what method of solving you will use prior to the start of this lesson – you can solve using excel, Gaussian Elimination, or TI-84 calculators. This is

			important, as the guided notes are different for each method of solving.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	N/A	N/A	N/A
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~30 mins	Students will break into (or be put into) groups of 3-4 students. They will them compete in Rock, Paper, Scissors to collect the data for the activity for the following day. The students should enter their win/loss stats into the Google Form (or by the method that you choose as the teacher).	Break students into 6-10 teams (about 3-4 students per team). Adjust this number according to the size of your class. Facilitate the data collection process. Be sure that the students are actively participating in data collection and are entering their data into the appropriate location. Consolidate the data into one convenient location in preparation for tomorrow's lesson.
Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	~ 10 mins	Complete the exit ticket (Lesson 4 – Colley's Method Problem Set)	Direct the students to the appropriate location to complete the exit ticket/quiz – you can have the students complete this on paper, using Google Forms, or through some other form of your choice

Video Discussion Questions

"How to Pick a Winning March Madness Bracket" Discussion Questions:

What steps do you need to complete to pick a winning bracket?

1. Read sports sites and blogs
    a. Bonus Question: What should you do when checking sports sites and blogs?
2. Take seeding into account
    a. Bonus Question: When should you start to ignore seedings of teams?
3. Check the rebound statistics
4. Study the offensive stats of each team
5. Focus on teams that can play zone and man-to-man defense
6. Try to figure out the picks of others and deviate when reasonable
7. When in doubt, go with your gut

(NOTE: The following guided notes are provided in three versions to address each of three approaches to applying the Method: 1) Excel; 2) Gaussian Elimination; and 3) TI-84 Calculator)

Guided Notes - Teacher Version – Excel

Introduction to Colley's Method Name: ______________________________

Date: ________________ Period: _______

Given a list of items:

- **Ranking:** an ordering of items
- **Rating:** assign a numerical score to each item

Examples of Rankings/Ratings:

- Sports: best teams in the league
- Schools: best schools in the nation
- Search results: being on the first page of a Google search
- Social networks: becoming an influencer

Key Challenges:

- **Objectivity:** process for determining ranking based on objective data
- **Transparency:** the simplicity of the system (is the system easy to understand?)
- **Robustness:** ability to withstand adverse conditions (make sure that the method doesn't include a means which is achievable without effort – example: if college teams are ranked by their win/loss record only, they could selectively make their schedule so that they play easy to beat teams)

Win/Loss Records: These are meant as talking points for a discussion with the students.

Can we use just the win/loss records to rank teams?

What are some challenges to considering only the win/loss records?

Considerations for win/loss records:

- How could you account for strength of schedule? What if teams try to play all easy-to-beat teams to earn a higher win/loss record?
- Should we take the margin of victory into account? What if the game is a close game? A blowout?
- Should there be correction for home/away games or other factors?

Colley's Method of Ranking:

Colley's Method of Ranking began as a slight modification to the general ranking based on win percentage. This method has its advantages because it does not rank based on just the win percentage, therefore the teams cannot build their schedules to play easy-to-beat teams and rack up their win percentage to rank higher. This method encourages the teams to play more difficult-to-beat teams, because if they beat those higher ranked teams, then they will earn more points to their own ranking. Colley's method also considers the win/loss record and the total number of games but uses this information differently. The first step is to construct an n x n matrix C, which we call Colley's matrix, and an n x 1 vector b. The next step is to solve the linear system of equations $Cr=b$ to obtain Colley's ratings r. Finally, we use the ratings vector to determine the rankings (higher values of r, means higher ranking). (Source: Who's #1 The Science of Rating and Ranking)

Variables and their Meanings

N: # of teams

w_i: # of wins for team i

l_i: # of losses for team i

t_i: # of games for team i

r_i: Colley Rating

n_{ij}: # of times team i played team j

- Matrix System:

$$(2+t_i)r_i - \sum_{j=1}^{N}(n_{ij}r_j) = 1 + \frac{w_i - l_i}{2}$$

To Solve: *Cr=b*

The number of times team 1 plays team 2

$$C = \begin{bmatrix} 2+t_1 & -n_{12} & -n_{13} & \cdots & -n_{1N} \\ -n_{21} & 2+t_2 & -n_{23} & \ddots & -n_{2N} \\ -n_{31} & -n_{32} & 2+t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -n_{N1} & \cdots & \cdots & \cdots & 2+t_N \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad b = \begin{bmatrix} 1+\dfrac{w_1-l_1}{2} \\ 1+\dfrac{w_2-l_2}{2} \\ \vdots \\ 1+\dfrac{w_N-l_N}{2} \end{bmatrix}$$

Examples:

1) College Football Records

	Duke	Miami	UNC	UVA	VT	Record
Duke		7-52	21-24	7-38	0-45	0-4
Miami	52-7		34-16	25-17	27-7	4-0
UNC	24-21	16-34		7-5	3-30	2-2
UVA	38-7	17-25	5-7		14-52	1-3
VT	45-0	2-27	30-3	52-14		3-1

$$t_i = 4 \qquad 2+t_i = 6$$

$$n_{ij} = 1 \qquad b = 1 + \frac{w-l}{2}$$

$$C = \begin{bmatrix} 6 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & 6 \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

In Excel: To Calculate C-Inverse: =MINVERSE(*array*) and To Calculate r: =MMULT(*C-Inverse array, r array*)

	C						b	
	6	-1	-1	-1	-1		-1	
	-1	6	-1	-1	-1		3	
	-1	-1	6	-1	-1		1	
	-1	-1	-1	6	-1		0	
	-1	-1	-1	-1	6		2	
							r	
C-inverse	0.2142857	0.071428571	0.071428571	0.071428571	0.071428571		0.21	Duke
	0.0714286	0.214285714	0.071428571	0.071428571	0.071428571		0.79	Miami
	0.0714286	0.071428571	0.214285714	0.071428571	0.071428571		0.50	Unc
	0.0714286	0.071428571	0.071428571	0.214285714	0.071428571		0.36	Uva
	0.0714286	0.071428571	0.071428571	0.071428571	0.214285714		0.64	Vt

2) ***YOU TRY!*** **Movie Ratings**

	Fargo	**Shrek**	**Milk**	**Jaws**
User 1	5	4	3	-
User 2	5	5	3	1
User 3	-	-	-	5
User 4	-	-	2	-
User 5	4	-	-	3
User 6	1	-	-	4

*A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for user 1, Fargo beats Shrek because a 5 is higher than a 4. You should compare all movies in this manner. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

Students may need some guidance with this.

	w_i	l_i	**Ties**	t_i	$t_i + 2$
Fargo	5	1	1	7	9
Shrek	3	1	1	5	7
Milk	1	4	0	5	7
Jaws	1	4	0	5	7

$$C = \begin{bmatrix} 9 & -2 & -2 & -3 \\ -2 & 7 & -2 & -1 \\ -2 & -2 & 7 & -1 \\ -3 & -1 & -1 & 7 \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

In Excel:

	C					b	
	9	-2	-2	-3		3	
	-2	7	-2	-1		2	
	-2	-2	7	-1		-0.5	
	-3	-1	-1	7		-0.5	
						r	
C-inverse	0.191860465	0.098837209	0.098837209	0.110465116		0.669	Fargo
	0.098837209	0.2125323	0.101421189	0.087209302		0.627	Shrek
	0.098837209	0.101421189	0.2125323	0.087209302		0.349	Milk
	0.110465116	0.087209302	0.087209302	0.215116279		0.355	Jaws

Guided Notes – Teacher Version – Gaussian Elimination

Introduction to Colley's Method

Name: ______________________________
Date: ________________ Period: _______

Given a list of items:

- **Ranking:** an ordering of items
- **Rating:** assign a numerical score to each item

Examples of Rankings/Ratings:

- Sports: best teams in the league
- Schools: best schools in the nation
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- **Objectivity:** process for determining ranking based on objective data
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Variables and their Meanings

N:	**# of teams**	w_i:	**# of wins for team i**
l_i:	**# of losses for team i**	t_i:	**# of games for team i**
r_i:	**Colley Rating**	n_{ij}:	**# of times team i played team j**

- **Matrix System:**

$$(2+t_i)r_i - \sum_{j=1}^{N}(n_{ij}r_j) = 1 + \frac{w_i - l_i}{2}$$

The number of times team 1 plays team 2

To Solve: *Cr=b*

$$C=\begin{bmatrix} 2+t_1 & -n_{12} & -n_{13} & \cdots & -n_{1N} \\ -n_{21} & 2+t_2 & -n_{23} & \ddots & -n_{2N} \\ -n_{31} & -n_{32} & 2+t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -n_{N1} & \cdots & \cdots & \cdots & 2+t_N \end{bmatrix} \quad r=\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad b=\begin{bmatrix} 1+\frac{w_1-l_1}{2} \\ 1+\frac{w_2-l_2}{2} \\ \vdots \\ 1+\frac{w_N-l_N}{2} \end{bmatrix}$$

Examples:

3) **College Football Records**

	Duke	Miami	UNC	UVA	VT	Record
Duke		7-52	21-24	7-38	0-45	0-4
Miami	52-7		34-16	25-17	27-7	4-0
UNC	24-21	16-34		7-5	3-30	2-2
UVA	38-7	17-25	5-7		14-52	1-3
VT	45-0	2-27	30-3	52-14		3-1

$$t_i = 4 \qquad 2+t_i = 6$$

$$n_{ij} = 1 \qquad b = 1+\frac{w-l}{2}$$

$$C=\begin{bmatrix} 6 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & 6 \end{bmatrix} \quad r=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \quad b=\begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & 3 \\ -1 & -1 & 6 & -1 & -1 & 1 \\ -1 & -1 & -1 & 6 & -1 & 0 \\ -1 & -1 & -1 & -1 & 6 & 2 \end{bmatrix}$$

Gaussian Elimination:

1) $$\left[\begin{array}{ccccc|c} 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & 3 \\ -1 & -1 & 6 & -1 & -1 & 1 \\ -1 & -1 & -1 & 6 & -1 & 0 \\ -1 & -1 & -1 & -1 & 6 & 2 \end{array}\right] \leftarrow R_3 \leftrightarrow R_1$$

2) $\begin{bmatrix} -1 & -1 & 6 & -1 & -1 & | & 1 \\ -1 & 6 & -1 & -1 & -1 & | & 3 \\ 6 & -1 & -1 & -1 & -1 & | & -1 \\ -1 & -1 & -1 & 6 & -1 & | & 0 \\ -1 & -1 & -1 & -1 & 6 & | & 2 \end{bmatrix} \leftarrow$ -R$_1$

3) $\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ -1 & 6 & -1 & -1 & -1 & | & 3 \\ 6 & -1 & -1 & -1 & -1 & | & -1 \\ -1 & -1 & -1 & 6 & -1 & | & 0 \\ -1 & -1 & -1 & -1 & 6 & | & 2 \end{bmatrix} \leftarrow$ R$_1$+R$_2$

4) $\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 6 & -1 & -1 & -1 & -1 & | & -1 \\ -1 & -1 & -1 & 6 & -1 & | & 0 \\ -1 & -1 & -1 & -1 & 6 & | & 2 \end{bmatrix} \leftarrow$ -6R$_1$+R$_3$

5) $\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & -7 & 35 & -7 & -7 & | & 5 \\ -1 & -1 & -1 & 6 & -1 & | & 0 \\ -1 & -1 & -1 & -1 & 6 & | & 2 \end{bmatrix} \leftarrow$ R$_1$+R$_4$

6) $\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & -7 & 35 & -7 & -7 & | & 5 \\ 0 & 0 & -7 & 7 & 0 & | & -1 \\ -1 & -1 & -1 & -1 & 6 & | & 2 \end{bmatrix} \leftarrow$ R$_1$+R$_5$

7) $\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & -7 & 35 & -7 & -7 & | & 5 \\ 0 & 0 & -7 & 7 & 0 & | & -1 \\ 0 & 0 & -7 & 0 & 7 & | & 1 \end{bmatrix} \leftarrow$ R$_2$+R$_3$

8) $\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & -7 & 7 & 0 & | & -1 \\ 0 & 0 & -7 & 0 & 7 & | & 1 \end{bmatrix} \leftarrow$ 4R$_4$

9) $$\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & -28 & 28 & 0 & | & -4 \\ 0 & 0 & -7 & 0 & 7 & | & 1 \end{bmatrix} \leftarrow R_3+R_4$$

10) $$\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & 0 & 21 & -7 & | & 3 \\ 0 & 0 & -7 & 0 & 7 & | & 1 \end{bmatrix} \leftarrow 4R_5$$

11) $$\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & 0 & 21 & -7 & | & 3 \\ 0 & 0 & -28 & 0 & 28 & | & 4 \end{bmatrix} \leftarrow R_3+R_5$$

12) $$\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & 0 & 21 & -7 & | & 3 \\ 0 & 0 & 0 & -7 & 21 & | & 11 \end{bmatrix} \leftarrow 3R_5$$

13) $$\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & 0 & 21 & -7 & | & 3 \\ 0 & 0 & 0 & -21 & 63 & | & 33 \end{bmatrix} \leftarrow R_4+R_5$$

14) $$\begin{bmatrix} 1 & 1 & -6 & 1 & 1 & | & -1 \\ 0 & 7 & -7 & 0 & 0 & | & 2 \\ 0 & 0 & 28 & -7 & -7 & | & 7 \\ 0 & 0 & 0 & 21 & -7 & | & 3 \\ 0 & 0 & 0 & 0 & 56 & | & 36 \end{bmatrix}$$

System of Equations:

$$\begin{array}{llllll} r_1 & +r_2 & -6r_3 & +r_4 & +r_5 & = -1 \\ & 7r_2 & -7r_3 & & & = 2 \\ & & 28r_3 & -7r_4 & -7r_5 & = 7 \\ & & & 21r_4 & -7r_5 & = 3 \\ & & & & 56r_5 & = 36 \end{array}$$

Solve for r_5:

$$56r_5 = 36$$
$$\div 56 \quad \div 56$$
$$r_5 = \frac{9}{14} = 0.64$$

Solve for r_4:

$$21r_4 - 7r_5 = 3$$
$$21r_4 - 7\left(\frac{9}{14}\right) = 3$$
$$21r_4 - \frac{9}{2} = 3$$
$$+\frac{9}{2} \quad +\frac{9}{2}$$
$$21r_4 = \frac{15}{2}$$
$$*\frac{1}{21} \quad *\frac{1}{21}$$
$$r_4 = \frac{5}{14} = 0.36$$

Solve for r_3:

$$28r_3 - 7r_4 - 7r_5 = 7$$
$$28r_3 - 7\left(\frac{5}{14}\right) - 7\left(\frac{9}{14}\right) = 7$$
$$28r_3 - \frac{5}{2} - \frac{9}{2} = 7$$
$$28r_3 - 7 = 7$$
$$+7 \quad +7$$
$$28r_3 = 14$$
$$\div 28 \quad \div 28$$
$$r_3 = \frac{1}{2} \quad 0.5$$

Solve for r_2:

$$7r_2 - 7r_3 = 2$$
$$7r_2 - 7\left(\frac{1}{2}\right) = 2$$
$$7r_2 - \frac{7}{2} = 2$$
$$+\frac{7}{2} \quad +\frac{7}{2}$$
$$7r_2 = \frac{11}{2}$$

$$*\frac{1}{7} \qquad\qquad *\frac{1}{7}$$

$$r_2 \qquad\qquad =\frac{11}{14} \quad 0.79$$

Solve for r_1:

$$\begin{array}{llllll} r_1 & +r_2 & -6r_3 & +r_4 & +r_5 & =-1 \\ r_1 & +\frac{11}{14} & -6\left(\frac{1}{2}\right) & +\frac{5}{14} & +\frac{9}{14} & =-1 \\ r_1 & & -\frac{17}{14} & & & =-1 \\ & & +\frac{17}{14} & & & +\frac{17}{14} \\ r_1 & & & & & =\frac{3}{14} \quad 0.21 \end{array}$$

Final Answer: $r=\begin{bmatrix}0.21\\0.79\\0.50\\0.36\\0.64\end{bmatrix} \rightarrow \begin{array}{c} Duke \\ Miami \\ UNC \\ UVA \\ VT \end{array}$

4) ***YOU TRY!*** **Movie Ratings**

	Fargo	Shrek	Milk	Jaws
User 1	5	4	3	-
User 2	5	5	3	1
User 3	-	-	-	5
User 4	-	-	2	-
User 5	4	-	-	3
User 6	1	-	-	4

*A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for user 1, Fargo beats Shrek because a 5 is higher than a 4. You should compare all movies in this manner. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

Student may need some guidance with this.

	w_i	l_i	Ties	t_i	t_i+2
Fargo	5	1	1	7	9
Shrek	3	1	1	5	7

Milk	1	4	0	5	7
Jaws	1	4	0	5	7

$$C = \begin{bmatrix} 9 & -2 & -2 & -3 \\ -2 & 7 & -2 & -1 \\ -2 & -2 & 7 & -1 \\ -3 & -1 & -1 & 7 \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 9 & -2 & -2 & -3 & 3 \\ -2 & 7 & -2 & -1 & 2 \\ -2 & -2 & 7 & -1 & -\frac{1}{2} \\ -3 & -1 & -1 & 7 & -\frac{1}{2} \end{bmatrix}$$

Gaussian Elimination:

1) $$\begin{bmatrix} 9 & -2 & -2 & -3 & | & 3 \\ -2 & 7 & -2 & -1 & | & 2 \\ -2 & -2 & 7 & -1 & | & -\frac{1}{2} \\ -3 & -1 & -1 & 7 & | & -\frac{1}{2} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow 4R_3 \\ \\ \end{matrix}$$

2) $$\begin{bmatrix} 9 & -2 & -2 & -3 & | & 3 \\ -2 & 7 & -2 & -1 & | & 2 \\ -8 & -8 & 28 & -4 & | & -2 \\ -3 & -1 & -1 & 7 & | & -\frac{1}{2} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow 2R_4 \\ \\ \end{matrix}$$

3) $$\begin{bmatrix} 9 & -2 & -2 & -3 & | & 3 \\ -2 & 7 & -2 & -1 & | & 2 \\ -8 & -8 & 28 & -4 & | & -2 \\ -6 & -2 & -2 & 14 & | & -1 \end{bmatrix} \leftarrow R_1+R_3$$

4) $$\begin{bmatrix} 1 & -10 & 26 & -7 & | & 1 \\ -2 & 7 & -2 & -1 & | & 2 \\ -8 & -8 & 28 & -4 & | & -2 \\ -6 & -2 & -2 & 14 & | & -1 \end{bmatrix} \leftarrow 2R_1+R_2$$

5) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & -13 & 50 & -15 & 4 \\ -8 & -8 & 28 & -4 & -2 \\ -6 & -2 & -2 & 14 & -1 \end{array}\right] \leftarrow$ $8R_1+R_3$

6) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & -13 & 50 & -15 & 4 \\ 0 & -88 & 236 & -60 & 6 \\ -6 & -2 & -2 & 14 & -1 \end{array}\right] \leftarrow$ $6R_1+R_4$

7) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & -13 & 50 & -15 & 4 \\ 0 & -88 & 236 & -60 & 6 \\ 0 & -62 & 154 & -28 & 5 \end{array}\right] \leftarrow$ $-R_2$

8) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & 13 & -50 & 15 & -4 \\ 0 & -88 & 236 & -60 & 6 \\ 0 & -62 & 154 & -28 & 5 \end{array}\right] \leftarrow$ $88R_2+13R_3$

9) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & 13 & -50 & 15 & -4 \\ 0 & 0 & -1332 & 540 & -274 \\ 0 & -62 & 154 & -28 & 5 \end{array}\right] \leftarrow$ $62R_2+13R_4$

10) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & 13 & -50 & 15 & -4 \\ 0 & 0 & -1332 & 540 & -274 \\ 0 & 0 & -1098 & 566 & -183 \end{array}\right] \leftarrow$ $-1/2\ R_3$

11) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & 13 & -50 & 15 & -4 \\ 0 & 0 & 666 & -270 & 137 \\ 0 & 0 & -1098 & 566 & -183 \end{array}\right] \leftarrow$ $1098R_3+666R_4$

12) $\left[\begin{array}{cccc|c} 1 & -10 & 26 & -7 & 1 \\ 0 & 13 & -50 & 15 & -4 \\ 0 & 0 & 666 & -270 & 137 \\ 0 & 0 & 0 & 80,496 & 28,548 \end{array}\right]$

Solve the System of Equations:

$$\begin{array}{lllll} r_1 & -10r_2 & +26r_3 & -6r_4 & = 1 \\ & 13r_2 & -50r_3 & +15r_4 & = -4 \\ & & 666r_3 & -270r_4 & = 137 \\ & & & 80496r_4 & = 28548 \end{array}$$

Solve for r_4:

$$\begin{array}{lll} 80496r_4 & = 28548 & \\ \div 80496 & \div 80496 & \\ r_4 & = \frac{61}{172} & 0.355 \end{array}$$

Solve for r_3:

$$\begin{array}{lll} 666r_3 & -270r_4 & = 137 \\ 666r_3 & -270\left(\frac{61}{172}\right) & = 137 \\ 666r_3 & -\frac{8235}{86} & = 137 \\ & +\frac{8235}{86} & +\frac{8235}{86} \\ 666r_3 & & = \frac{12623}{1548} \\ *\frac{1}{666} & & *\frac{1}{666} \\ r_3 & & = \frac{541}{1548} \quad 0.349 \end{array}$$

Solve for r_2:

$$\begin{array}{llll} 13r_2 & -50r_3 & +15r_4 & = -4 \\ 13r_2 & -50\left(\frac{541}{1548}\right) & +15\left(\frac{16}{172}\right) & = -4 \\ 13r_2 & -\frac{18815}{1548} & & = -4 \\ & +\frac{18815}{1548} & & +\frac{18815}{1548} \\ 13r_2 & & & = \frac{12623}{1548} \\ *\frac{1}{13} & & & *\frac{1}{13} \\ r_2 & & & = \frac{971}{1548} \quad = 0.627 \end{array}$$

Solve for r_1:

$$\begin{array}{lllll} r_1 & -10r_2 & +26r_3 & -6r_4 & = 1 \\ r_1 & -10\left(\frac{971}{1548}\right) & +26\left(\frac{541}{1548}\right) & -7\left(\frac{61}{172}\right) & = 1 \end{array}$$

$$r_1 + \frac{57}{172} = 1$$

$$-\frac{57}{172} \quad -\frac{57}{172}$$

$$r_1 = \frac{115}{172} \quad 0.669$$

Final Answer: $r = \begin{bmatrix} 0.669 \\ 0.627 \\ 0.349 \\ 0.355 \end{bmatrix} \rightarrow \begin{matrix} Fargo \\ Shrek \\ Milk \\ Jaws \end{matrix}$

Guided Notes – Teacher Version – TI84

Introduction to Colley's Method

Name: ____________________________

Date: _______________ Period: ______

Given a list of items:

- **Ranking:** an ordering of items
- **Rating:** assign a numerical score to each item

Examples of Rankings/Ratings:

- Sports: best teams in the league
- Schools: best schools in the nation
- Search results: being on the first page of a Google search
- Social networks: becoming an influencer

Key Challenges:

- **Objectivity:** process for determining ranking based on objective data
- **Transparency:** the simplicity of the system (is the system easy to understand?)
- **Robustness:** ability to withstand adverse conditions (make sure that the method doesn't include a means which is achievable without effort – example: if college teams are ranked by their win/loss record only, they could selectively make their schedule so that they play easy to beat teams)

Win/Loss Records: These are meant as talking points for a discussion with the students.

Can we use just the win/loss records to rank teams?

What are some challenges to considering only the win/loss records?

Considerations for win/loss records:

- How could you account for strength of schedule? What if teams try to play all easy-to-beat teams to earn a higher win/loss record?

- Should we take the margin of victory into account? What if the game is a close game? A blowout?
- Should there be correction for home/away games or other factors?

Colley's Method of Ranking:

Colley's Method of Ranking began as a slight modification to the general ranking based on win percentage. This method has its advantages because it does not rank based on just the win percentage, therefore the teams cannot build their schedules to play easy-to-beat teams and rack up their win percentage to rank higher. This method encourages the teams to play more difficult-to-beat teams, because if they beat those higher ranked teams, then they will earn more points to their own ranking. Colley's method also considers the win/loss record and the total number of games but uses this information differently. The first step is to construct an n x n matrix *C*, which we call Colley's matrix, and an n x 1 vector *b*. The next step is to solve the linear system of equations *Cr=b* to obtain Colley's ratings *r*. Finally, we use the ratings vector to determine the rankings (higher values of *r*, means higher ranking). (Source: Who's #1 The Science of Rating and Ranking)

Variables and their Meanings

N:	**# of teams**	w_i:	**# of wins for team i**
l_i:	**# of losses for team i**	t_i:	**# of games for team i**
r_i:	**Colley Rating**	n_{ij}:	**# of times team i played team j**

- **Matrix System:**

$$(2+t_i)r_i - \sum_{j=1}^{N}(n_{ij}r_j) = 1 + \frac{w_i - l_i}{2}$$

To Solve: ***Cr=b***

The number of times team 1 plays team 2

$$C = \begin{bmatrix} 2+t_1 & -n_{12} & -n_{13} & \cdots & -n_{1N} \\ -n_{21} & 2+t_2 & -n_{23} & \ddots & -n_{2N} \\ -n_{31} & -n_{32} & 2+t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -n_{N1} & \cdots & \cdots & \cdots & 2+t_N \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad b = \begin{bmatrix} 1+\dfrac{w_1-l_1}{2} \\ 1+\dfrac{w_2-l_2}{2} \\ \vdots \\ 1+\dfrac{w_N-l_N}{2} \end{bmatrix}$$

Examples:

5) **College Football Records**

	Duke	Miami	UNC	UVA	VT	Record

Duke		7-52	21-24	7-38	0-45	0-4
Miami	52-7		34-16	25-17	27-7	4-0
UNC	24-21	16-34		7-5	3-30	2-2
UVA	38-7	17-25	5-7		14-52	1-3
VT	45-0	2-27	30-3	52-14		3-1

$$\boldsymbol{t_i = 4} \qquad\qquad \boldsymbol{2 + t_i = 6}$$

$$\boldsymbol{n_{ij} = 1} \qquad\qquad \boldsymbol{b = 1 + \frac{w - l}{2}}$$

$$\boldsymbol{C = \begin{bmatrix} 6 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 \\ -1 & -1 & 6 & -1 & -1 \\ -1 & -1 & -1 & 6 & -1 \\ -1 & -1 & -1 & -1 & 6 \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}}$$

Write the Augmented Matrix:

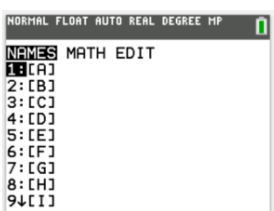
$$A = \begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & 3 \\ -1 & -1 & 6 & -1 & -1 & 1 \\ -1 & -1 & -1 & 6 & -1 & 0 \\ -1 & -1 & -1 & -1 & 6 & 2 \end{bmatrix}$$

Solve for the *"r"* matrix using your TI-84 Calculator:

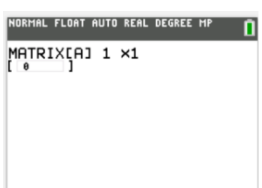
$$r = \begin{bmatrix} 0.21 \\ 0.79 \\ 0.50 \\ 0.36 \\ 0.64 \end{bmatrix}$$

For teacher use – How to find the *"r"* matrix using

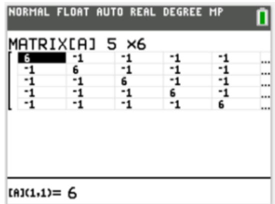
1. **2nd matrix:**



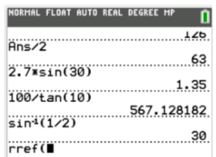
2. **Go to EDIT and choose 1: [A]**



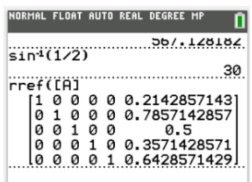
3. **Change the dimensions to the dimensions of the augmented matrix and then enter numbers**



4. **2nd quit → 2nd matrix → MATH → B: rref(→ ENTER**



5. **2nd matrix → 1:[A] → ENTER → ENTER**



6) ***YOU TRY!*** **Movie Ratings**

	Fargo	Shrek	Milk	Jaws
User 1	5	4	3	-
User 2	5	5	3	1
User 3	-	-	-	5
User 4	-	-	2	-
User 5	4	-	-	3
User 6	1	-	-	4

*A movie “wins” if it has a higher rating than the other movie it is “competing” against, i.e. for user 1, Fargo beats Shrek because a 5 is higher than a 4. You should compare all movies in this manner. If a movie does not have a rating, then it is not competing in that “round”. If there is a tie, then it does not count as a win or a loss.

Students may need some guidance with this

	w_i	l_i	**Ties**	t_i	$t_i + 2$
Fargo	5	1	1	7	9
Shrek	3	1	1	5	7

Milk	1	4	0	5	7
Jaws	1	4	0	5	7

$$C = \begin{bmatrix} 9 & -2 & -2 & -3 \\ -2 & 7 & -2 & -1 \\ -2 & -2 & 7 & -1 \\ -3 & -1 & -1 & 7 \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 2 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Write the Augmented Matrix:

$$A = \begin{bmatrix} 9 & -2 & -2 & -3 & 3 \\ -2 & 7 & -2 & -1 & 2 \\ -2 & -2 & 7 & -1 & -\frac{1}{2} \\ -3 & -1 & -1 & 7 & -\frac{1}{2} \end{bmatrix}$$

Solve for the *"r"* matrix using your TI-84 Calculator:

$$r = \begin{bmatrix} 0.669 \\ 0.627 \\ 0.349 \\ 0.355 \end{bmatrix}$$

Colley's Method Problem Set – Teacher Version

Colley's Method Problem Set

At the Movies

Five friends rate five different movies on a scale of 1 to 5. They do not know each other's ratings, and some of them have not seen all of the movies. A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for Madison, Avengers: Endgame beats Toy Story 4, since she rated the former a 4 and the latter a 3. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

Movie Title/ Rating	LOTR: Return of the King	Star Wars	Toy Story 4	Harry Potter and the Sorcerer's Stone	Avengers: Endgame
Madison	5	3	3	2	4
Kelia	4	4	--	4	2
Raffi	2	--	3	1	5

Rachel	5	2	4	2	--
Owen	3	5	4	--	--

1. Complete the following table given the ratings above.

i	**Movie i**	**# Wins**	**# Losses**	**# Ties**	**# of Comparisons**	$b_i = 1 + \frac{w_i - l_i}{2}$
1	LOTR: Return of the King	9	4	2	15	7/2
2	Star Wars	4	4	4	12	1
3	Toy Story 4	6	5	1	12	3/2
4	Harry Potter and the Sorcerer's Stone	1	9	3	13	-3
5	Avengers: Endgame	6	4	0	10	2

2. Write the Colley Matrix in the matrix equation and the vector on the right ("***b***" vector) that are associated with the information above.

$$\boldsymbol{C} = \begin{bmatrix} 17 & -4 & -4 & -4 & -3 \\ -4 & 14 & -3 & -3 & -2 \\ -4 & -3 & 14 & -3 & -2 \\ -4 & -3 & -3 & 15 & -3 \\ -3 & -2 & -2 & -3 & 12 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 7/2 \\ 1 \\ 3/2 \\ -3 \\ 2 \end{bmatrix}$$

3. Solve for the ratings using technology, and convert to the Colley ranking.

$$\boldsymbol{r} = \begin{bmatrix} 0.62 \\ 0.50 \\ 0.53 \\ 0.28 \\ 0.56 \end{bmatrix}$$

i	**Movie i**	**Colley Rank**
1	LOTR: Return of the King	1
2	Star Wars	4

3	Toy Story 4	**3**
4	Harry Potter	**5**
5	Avengers: Endgame	**2**

Colley's Method NCAA Division Basketball Problem[3]

The following is data from the games played in the America East conference from January 2, 2013, to January 10, 2013 in the 2013 NCAA Men's Division 1 Basketball. (This data can be found on the ESPN website.)

The teams in the conference are as follows:

i	Team i	Abbreviation
1	Stony Brook	STON
2	Vermont	UVM
3	Boston University	BU
4	Hartford	HART
5	Albany	ALBY
6	Maine	ME
7	Univ. Maryland, Bal. County	UMBC
8	New Hampshire	UNH
9	Binghampton	BING

The following is a record of their games and results (W/L) from January 2, 2013, to January 10, 2013:

Date	**Teams**	**Winner**
Jan 02, 2013	BING vs HART	HART
Jan 02, 2013	UVM vs UNH	UVM
Jan 02, 2013	BU vs ME	ME
Jan 02, 2013	ALBY vs UMBC	ALBY
Jan 05, 2013	STON vs UNH	STON
Jan 05, 2013	UVM vs ALBY	UVM
Jan 05, 2013	BU vs HART	HART
Jan 05, 2013	ME vs UMBC	ME
Jan 07, 2013	BING vs ALBY	ALBY
Jan 08, 2013	UVM vs BU	BU

[3] Source: https://www3.nd.edu/~apilking/Math10170/Information/Lectures%202015/Topic8Colley.pdf

Jan 09, 2013	BING vs STON	STON
Jan 09, 2013	ME vs HART	HART
Jan 09, 2013	UMBC vs UNH	UMBC

1. Complete the following table given the information above.

i	Team i	Abbrev	# Wins	# Losses	# Ties	# of Comparisons	$b_i = 1 + \frac{w_i - l_i}{2}$
1	Stony Brook	STON	2	0	0	2	2
2	Vermont	UVM	2	1	0	3	3/2
3	Boston University	BU	1	2	0	3	1/2
4	Hartford	HART	3	0	0	3	5/2
5	Albany	ALBY	2	1	0	3	3/2
6	Maine	ME	2	1	0	3	3/2
7	Univ. Maryland, Bal. County	UMBC	1	2	0	3	1/2
8	New Hampshire	UNH	0	3	0	3	-1/2
9	Binghampton	BING	0	3	0	3	-1/2

2. Write the Colley Matrix in the matrix equation and the vector on the right ("$\boldsymbol{b}$" vector) that are associated with the information above.

$$\boldsymbol{C} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 5 & -1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 5 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 5 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 0 & 5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 5 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 5 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3/2 \\ 1/2 \\ 5/2 \\ 3/2 \\ 3/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

3. Solve for the ratings using technology, and convert to the Colley rankings.

$$\boldsymbol{r} = \begin{bmatrix} \mathbf{0.625} \\ \mathbf{0.549} \\ \mathbf{0.492} \\ \mathbf{0.783} \\ \mathbf{0.543} \\ \mathbf{0.630} \\ \mathbf{0.377} \\ \mathbf{0.210} \\ \mathbf{0.290} \end{bmatrix}$$

i	Team i	Abbreviation	Colley Rank
1	Stony Brook	STON	**3**
2	Vermont	UVM	**4**
3	Boston University	BU	**6**
4	Hartford	HART	**1**
5	Albany	ALBY	**5**
6	Maine	ME	**2**
7	Univ. Maryland, Bal. County	UMBC	**7**
8	New Hampshire	UNH	**9**
9	Binghampton	BING	**8**

Lesson 5: Rock, Paper, Scissors Activity

Lesson Plan

Standard	Topic/Day
Standard : **HSN.VM.C.6** - Number and Quantity: Vector & Matrix Quantities Use matrices to represent and manipulate data. **HSN.VM.C.11.A** - Number and Quantity: Vector & Matrix Quantities Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Mathematical Practices: 1, 3, 5, and 7	**Topic/Day: Applying Colley's Method with Rock/Paper/Scissors** **Content Objective: Ranking and Matrices** **Materials Needed: Google Form, Excel Spreadsheet, Video and/or Article, Guided notes, Exit Ticket, Student and Teacher Handout with Steps for completing the Rock/Paper/Scissors activity**

	Time	Student Does	Teacher Does
Warm Up Elicit/Engage Build relevance through a problem Try to find out what your students already know Get them interested	10-15 min	Jigsaw article to encourage discussion and interest in ranking. https://www.forbes.com/sites/christinasettimi/2016/03/14/best-ways-to-pick-a-winning-ncaa-tournament-bracket/#6fac5a611687 Video options for students who do not like to read: https://www.youtube.com/watch?v=GX_3h5dap9s https://www.youtube.com/watch?v=V4LFxAO9dMw	Provide article and/or play videos and lead discussion.

Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	30 min	Using data collected in the previous class discuss and devise a way to organize the data to see who was the "best".	Listen to group conversations and ask questions to help guide and continue the discussions.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	30 min	Guided notes on the example provided by the teacher so they have the steps available for finding the rank.	Teacher will provide an example of ranking using ACC Basketball team records
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	15 min	If a group finishes before other groups encourage them to combine results of 2 groups to see if the ranking is drastically different when the group grows.	Ask guiding questions to encourage inquiry-based learning during this extension.
Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	15 min	Solve the exit ticket and give to teacher either on paper or virtually to allow for reteaching.	Create a simple example that students can "solve".

Steps for Rock, Paper, Scissors Activity – Teacher Version

Rock Paper Scissors



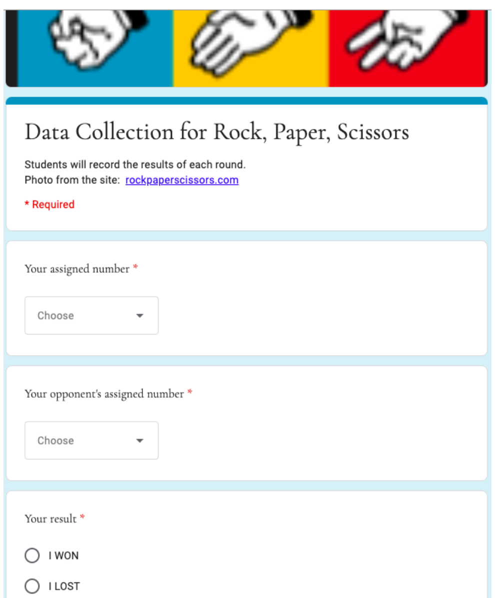
Day 1 (end of Lesson 4):

Step 1: Form groups of 4-5 students.

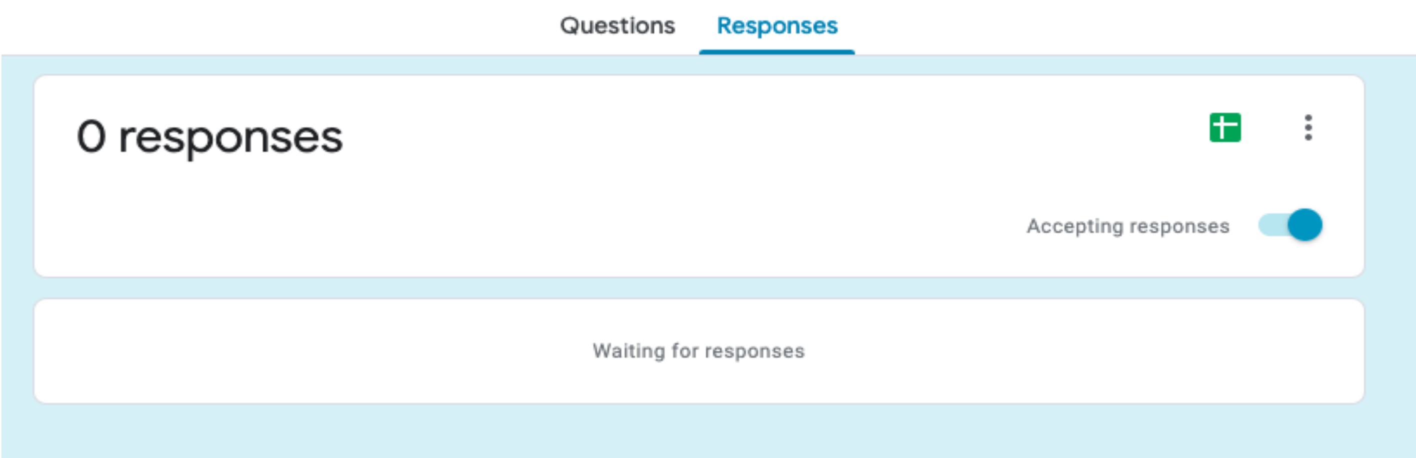
Step 2: Students will play "Rock, Paper, Scissors" with each person in the group playing every other person in the group. To determine a winner students should play 3 rounds with each partner and use best 2 out of 3.

Day 2 (Lesson 5):

Step 3: Students need to record who won each round in an organized way. If you want to use the Google Form have students assigned a number from 1-5 to identify which player they are. This was done to keep you from having to type in students names each time you change groups or classes. You can edit the form https://docs.google.com/forms/d/1RtvKYQ7TU5DgI6OQWFtml8-XNyk3GkvW4Y3T590tCnY/edit#responses
by making a copy and adding or deleting numbers as you need for the size of your class.
https://forms.gle/QLVAYYafNE6ZmtzRA The form looks like the image below when your students go to the link



Step 4: Once the students have collected the data to create a spreadsheet from the Google Form, the Teacher will convert the data into a Spreadsheet or organize it for calculations by hand (up to the discretion of the teacher). To create the spreadsheet from the Google Form choose "Responses" at the top and then choose the Green Box with the white cross inside as shown below.



Below is an example of what the spreadsheet will look like once responses are collected. The actual spreadsheet will be attached at the end of this unit so you can utilize the formulas in the appropriate cells.

C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
Your opponent's assigned	Your result		1	2	3	4	5		C						b	
1	I WON		-1	0	1	0	0		5	-1	0	-1	-1		1.5	This is the Colley matrix and the right hand side
2	I LOST		0	0	0	0	0		-1	6	-1	-1	-1		2	
5	I WON		0	0	0	1	-1		0	-1	5	-1	-1		0.5	
1	I LOST		0	0	0	0	0		-1	-1	-1	6	-1		0	
3	I LOST		0	0	0	0	0		-1	-1	-1	-1	6		1	
5	I WON		1	0	0	0	-1									
1	I WON		-1	1	0	0	0		C^{-1}						r	This is the inverse and the Colley rating
4	I LOST		0	0	0	0	0		0.24	0.07	0.04	0.07	0.07		0.60	
1	I LOST		0	0	0	0	0		0.07	0.21	0.07	0.07	0.07		0.64	
4	I WON		1	0	0	-1	0		0.04	0.07	0.24	0.07	0.07		0.40	
2	I LOST		0	0	0	0	0		0.07	0.07	0.07	0.21	0.07		0.36	
3	I WON		0	1	-1	0	0		0.07	0.07	0.07	0.07	0.21		0.50	
2	I LOST		0	0	0	0	0									
4	I WON		0	1	0	-1	0		b^T							
2	I WON		0	-1	0	0	1		1.5	2	0.5	0	1			
5	I LOST		0	0	0	0	0									
3	I LOST		0	0	0	0	0									
4	I WON		0	0	1	-1	0									
3	I WON		0	0	-1	0	1									
5	I LOST		0	0	0	0	0									

Step 5: Use the Colley's Method to determine who is the Rock Paper Scissors Champion.

Lesson 6: Final Assessment Project

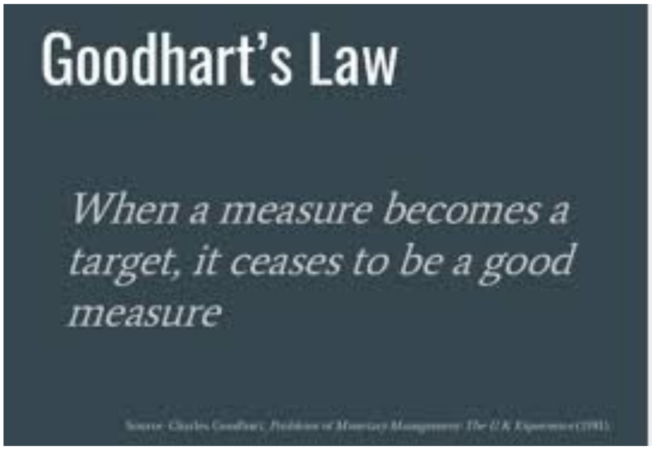
Colley's Method Final Project Lesson Plan

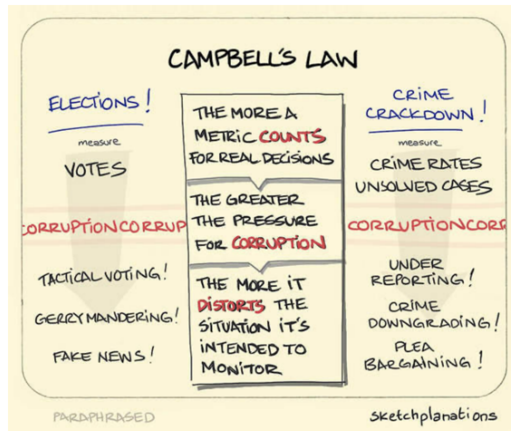
Standards: N/A	**Topic/Day: Performance Assessment (Project)** **Content Objective:** **Materials Needed: Graphic or Video of Goodhart's Law or Campbell's Law, Project instruction sheet** **(1-2 days)**

	Time	Student Does	Teacher Does
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	10 min	Write Noticing and Wondering statements about Goodhart's Law and/or Campbell's Law.	Provide Goodhart's Law quote and/or Campbell's Law quote on screen/board.
Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	45 min	Look for other topics where ranking is used or could be used. This can be something where the data has already been collected or a situation where the student devises a collection method.	Provide suggestions or options: NFL Teams, Movies, Basketball, Video Games, Social Media, Google Search, College Rankings, Music, Crime are just a few examples.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	90 min	Develop a plan, execute the plan to collect, organize (matrix) and analyze the data (Colley's method)	Be available to ask questions and help students who are "stuck".
Evaluate **Assessment** How will you know if students understand throughout the lesson?	As needed	Have students present their findings in a creative way- with technology or poster	Using a rubric decide if students understand the Colley's method and understand the process for

		Discuss the possible flaws in the system based on the results in your situation.	ranking in the application they chose. This project is the final assessment for this unit.

Goodhart's and Campbell's Laws Images and Discussion Questions





https://www.youtube.com/watch?v=SwcK7NI_i98 https://www.youtube.com/watch?v=OtWtld5ILOk

- Campbell's law: "The more any quantitative social indicator is used for social decision-making, the more subject it will be to corruption pressures and the more apt it will be to distort and corrupt the social processes it is intended to monitor" (social scientist and psychologist Donald T. Campbell)

- Goodhart's law: "When a measure becomes a target, it ceases to be a good measure" (named after economist Charles Goodhart)

  https://www.lesswrong.com/posts/YtvZxRpZjcFNwJecS/the-importance-of-goodhart-s-law#:~:text=The%20most%20famous%20examples%20of,a%20pre%2Dcentral%20plan%20scenario.

  https://thehustle.co/Goodharts-Law

1. If you were told exactly what the teacher is going to grade you on, how would that affect what you focus on?
2. If you are not graded on an assignment, how much effort will you put into it?
3. If you are told you are graded on the number of assignments you turn in, what would happen?
4. If you are graded on how long your answers are, how would that determine your focus?

5. If colleges solely chose students based on SAT scores, how would that change how students focused their attention?
6. When we think of test scores, what are the consequences (past, present or future) of putting value on scores of students, classes, and schools?

Colley's Method Final Project – Teacher Version

Final Project for Ranking with Colley's Method Teacher Notes

Day 1 (End of Lesson 5):

Step 1: Choose a topic you are interested in ranking.
Teacher notes: You may want to group students based on their interest.

Step 2: Either collect data or find data on this topic. For example if you are interested in the NFL you may want to use statistics that are already available on their site. However, if you want to rank music you may want to choose 5 songs and have others rate them and perform Colley's method on those 5 songs. Your choice!

Teacher notes: Here are a few sites to get your students started.

- Sports teams. College football teams are ranked by their BCS (Bowl Championship Series) rating, which helps determine which teams are invited to which bowl games. [1] Similarly, college basketball teams are ranked by their RPI (Rating Percentage Index), which determines which teams are invited to the March Madness tournament.
  - https://www.teamrankings.com/
- Individual athletes/competitors. FIDE (the international chess federation) uses the Elo system to rank chess players worldwide (also for some video/board games).
  - https://www.esportsearnings.com/players
- Colleges, hospitals, law schools, etc. Notably, the US News and World report ranking for colleges.

- https://www.usnews.com/best-colleges
- https://health.usnews.com/best-hospitals

- Search results (Google etc). Whether your business is the top hit on Google (or on the first page of results) can be a life-or-death matter depending on the business.
  - https://ahrefs.com/blog/most-visited-websites/
- Netflix, IMDB: Movie rankings, and more notably recommendation systems.
  - https://www.hollywoodreporter.com/lists/100-best-films-ever-hollywood-favorites-818512
- Human Development Index: Rank countries by education/literacy/standard of living. Used to decide how to allocate aid to underdeveloped countries.
  - https://www.usnews.com/news/best-countries/overall-rankings
- Social networks? Given a network of people, who is the most popular?
  - https://datareportal.com/social-media-users

Day 2 (Lesson 6):

Step 3: Organize your data in a table.

Step 4: Perform the steps you learned in the previous lesson using the Colley's Method for Ranking. (This can be done by hand or with Excel depending on your teacher's preference.)
Teacher notes: The various application options are provided in the guided notes.

Step 5: Using your results conclude who/what is the best and defend this assertion with the data.

Step 6: Display your results in a way that your classmates can easily understand.

Step 7: Consider any drawbacks or limitations this method had on your data. Are there flaws (drawbacks) with this ranking system? If so, what would you suggest as an alternative?

Advantages of Colley's Method	Drawbacks of Colley's Method and any ranking method
No bias toward conference, tradition or history	Any ranking system is subject to Campbell's Law and Goodhart's Law
It is reproducible	Is it simple enough to explain to others?
Uses a minimum of assumptions	Ties in the ratings often occur and must be dealt with fairly.

It uses a minimum amount of ad hoc adjustments	The reputation of the opponent is not factored in the analysis. (Only win/loss outcomes)
It adjusts for strength of schedule	The scores (close game or blow out) are not considered
Ignores runaway scores	Could argue that other factors are not considered or weighted fairly.
Produces common sense results that compare well to the press polls	Outside factors are not considered which could be unfair- culture, injuries, weather, natural disasters, pandemics, etc

References and Additional Readings

Bartkovich, K. G., Goebel, J. A., Graves, J. L., Teague, D. J., Barrett, G. B., Compton, H. L., & Whitehead, K. (2000). *Contemporary Precalculus through Applications*. New York: Glencoe/McGraw-Hill

Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

Matrix Methods

- Who's #1? The Science of Rating and Ranking. Amy Langville and Carl Meyer. https://press.princeton.edu/books/hardcover/9780691154220/whos-1
- Colley's Bias Free College Football Ranking Method: The Colley Ma-trix Explained. Wesley N. Colley. https://www.colleyrankings.com/matrate.pdf
- D1 Football rankings for 2019 season. https://www.colleyrankings. com/currank.html

Distortions

- Goodhart's Law. https://en.wikipedia.org/wiki/Goodhart%27s_law
- Campbell's Law. https://en.wikipedia.org/wiki/Campbell%27s_law

College Lists

- Methodology for 2020 College Rankings. US News & World Report. https://www.usnews.com/education/best-colleges/articles/how-us-news-calculated-the-rankings
- "U.S. News changed the way it ranks colleges. It's still ridiculous." Valerie Strauss, Washington Post, Sept 12, 2018. https://www.washingtonpost. com/education/2018/09/12/us-news-changed-way-it-ranks-colleges-its-still-ridiculous/
- "Is There Life After Rankings?" Colin Diver (president of Reed College), Atlantic Nov 2005 issue.

Movie Resources

- IMDB does not use the raw average of its user ratings. https://help.imdb.com/article/imdb/track-movies-tv/the-vote-average-for-film-x-should-be-y-why-are-you-displaying-another-rating/G3RC8ZNFAGWNTX4L?ref_=helpart_nav_9#
- IMDB ratings for Ghostbusters (2016). https://www.imdb.com/title/tt1289401/ratings?ref_=tt_ov_rt
- Rotten Tomatoes Top 100. https://www.rottentomatoes.com/top/bestofrt/
- Metacritic Top Movies. https://www.metacritic.com/browse/movies/score/metascore/all/filtered?sort=desc
- IMDB Top Rated Movies. https://www.imdb.com/chart/top/?ref_=nv_mv_250

Appendix: Lesson Materials – Student Versions

Lesson 1 - Guided Notes - Student Version

*Student Version begins on the next page.

Matrix Addition, Subtraction and Scalar Multiplication

A university is taking inventory of the books they carry at their two biggest bookstores.
The East Campus bookstore carries the following books:

Hardcover: Textbooks-5280; Fiction-1680; NonFiction-2320; Reference-1890

Paperback: Textbooks-1930; Fiction-2705; NonFiction-1560; Reference-2130

The West Campus bookstore carries the following books:

Hardcover: Textbooks-7230; Fiction-2450; NonFiction-3100; Reference-1380

Paperback: Textbooks-1740; Fiction-2420; NonFiction-1750; Reference-1170

In order to work with this information, we can represent the inventory of each bookstore using an organized array of numbers known as a *matrix*.

Definitions: A __________ is a rectangular table of entries and is used to organize data in a way that can be used to solve problems. The following is a list of terms used to describe matrices:

- A matrix's ___________________________ is written by listing the number of rows "by" the number of columns.
- The values in a matrix, A, are referred to as ______________ or ______________. The entry in the "m^{th}" row and "n^{th}" column is written as a_{mn}.
- A matrix is ______________ if it has the same number of rows as it has columns.
- If a matrix has only one row, then it is a row ____________. If it has only one column, then the matrix is a column ______________.
- The __________________ of a matrix, A, written A^T, switches the rows with the columns of A and the columns with the rows.
- Two matrices are _____________ if they have the same size and the same corresponding entries.

The inventory of the books at the East Campus bookstore can be represented with the following **2 x 4** matrix:

$$E = \begin{array}{r} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Similarly, the West Campus bookstore's inventory can be represented with the following matrix:

$$W = \begin{array}{r} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Adding and Subtracting Matrices

In order to add or subtract matrices, they must first be of the same ________________________.
The result of the addition or subtraction is a matrix of the same size as the matrices themselves, and the entries are obtained by adding or subtracting the elements in corresponding positions.

In our campus bookstores example, we can find the total inventory between the two bookstores as follows:

$$E + W = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} + \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{array}{r} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Question: *Is matrix addition commutative* $(e.g., A + B = B + A)$*? Why or why not?*

Question: *Is matrix subtraction commutative* $(e.g., A - B = B - A)$*? Why or why not?*

Question: *Is matrix addition associative* $(e.g., (A + B) + C = A + (B + C))$*? Why or why not?*

Question: *Is matrix subtraction associative* $(e.g., (A - B) - C = A - (B - C))$*? Why or why not?*

Scalar Multiplication

Multiplying a matrix by a constant (or *scalar*) is as simple as multiplying each entry by that number! Suppose the bookstore manager in East Campus wants to double his inventory. He can find the number of books of each type that he would need by simply multiplying the matrix E by the scalar (or constant) 2. The result is as follows:

$$2E = 2 * \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \end{bmatrix} \end{array}$$

Exercises: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed:

b. $A + B$

b. $B - A$

$c.$ $A - C$

d. $C - A$

e. $5B$

f. $-A + 4C$

g. $B - D$

h. $2C - 6A$

i. $B^T + D$

Lesson 2 - Guided Notes - Student Version

*Student Version begins on the next page.

Matrix Multiplication

The Metropolitan Opera is planning its last cross-country tour. It plans to perform *Carmen* and *La Traviata* in Atlanta in May. The person in charge of logistics wants to make plane reservations for the two troupes. *Carmen* has 2 stars, 25 other adults, 5 children, and 5 staff members. *La Traviata* has 3 stars, 15 other adults, and 4 staff members. There are 3 airlines to choose from. Redwing charges round-trip fares to Atlanta of $630 for first class, $420 for coach, and $250 for youth. Southeastern charges $650 for first class, $350 for coach, and $275 for youth. Air Atlanta charges $700 for first class, $370 for coach, and $150 for youth. Assume stars travel first class, other adults and staff travel coach, and children travel for the youth fare.

Use multiplication and addition to find the total cost for each troupe to travel each of the airlines.

It turns out that we can solve problems like these using a matrix operation, specifically **matrix multiplication**!

We first note that matrix multiplication is only defined for matrices of certain sizes. For the product AB of matrices A and B, where A is an $m\ x\ n$ matrix, B must have the same number of rows as A has columns. So, B must have size _____ $x\ p$. The product AB will have size _____________.

Exercises

The following is a set of abstract matrices (without row and column labels):

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad O = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 5 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

List at least 5 orders of pairs of matrices from this set for which the product is defined. State the dimension of each product.

Back to the opera…

Define two matrices that organize the information given:

$$\begin{array}{rc} & \begin{matrix}\textbf{stars} & \textbf{adults} & \textbf{children}\end{matrix} \\ \begin{matrix}\textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}}\end{matrix} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array} \qquad \begin{array}{rc} & \begin{matrix}\textbf{Red} & \textbf{South} & \textbf{Air}\end{matrix} \\ \begin{matrix}\textbf{stars} \\ \textbf{adults} \\ \textbf{children}\end{matrix} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array}$$

We can multiply these two matrices to obtain the same answers we obtained above, all in one matrix!

$$\begin{array}{rc} & \begin{matrix}\textbf{stars} & \textbf{adults} & \textbf{children}\end{matrix} \\ \begin{matrix}\textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}}\end{matrix} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array} \cdot \begin{array}{rc} & \begin{matrix}\textbf{Red} & \textbf{South} & \textbf{Air}\end{matrix} \\ \begin{matrix}\textbf{stars} \\ \textbf{adults} \\ \textbf{children}\end{matrix} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\end{array}$$

$$= \begin{array}{rc} & \begin{matrix}\textbf{Red} & \qquad\textbf{South}\qquad & \textbf{Air}\end{matrix} \\ \begin{matrix}\textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}}\end{matrix} & \begin{bmatrix} \qquad & \qquad & \qquad \\ \qquad & \qquad & \qquad \end{bmatrix}\end{array}$$

Carmen/Redwing:

Carmen/Southeastern:

Carmen/Air Atlanta:

La Traviata/Redwing:

La Traviata/Southeastern:

La Traviata/Air Atlanta:

Exercises[4]

3. The K.L. Mutton Company has investments in three states - North Carolina, North Dakota, and New Mexico. Its deposits in each state are divided among bonds, mortgages, and consumer loans. The amount of money (in millions of dollars) invested in each category on June 1 is displayed in the table below.

	NC	ND	NM
Bonds	13	25	22
Mort.	6	9	4
Loans	29	17	13

The current yields on these investments are 7.5% for bonds, 11.25% for mortgages, and 6% for consumer loans. Use matrix multiplication to find the total earnings for each state.

4. Several years ago, Ms. Allen invested in growth stocks, which she hoped would increase in value over time. She bought 100 shares of stock A, 200 shares of stock B, and 150 shares of stock C. At the end of each year she records the value of each stock. The table below shows the price per share (in dollars) of stocks A, B, and C at the end of the years 1984, 1985, and 1986.

	1984	1985	1986
Stock A	68.00	72.00	75.00
Stock B	55.00	60.00	67.50
Stock C	82.50	84.00	87.00

Calculate the total value of Ms. Allen's stocks at the end of each year.

[4] Adapted from Bartkovich, K. G., Goebel, J. A., Graves, J. L., Teague, D. J., Barrett, G. B., Compton, H. L., ... & Whitehead, K. (2000). *Contemporary Precalculus through Applications*. New York: Glencoe/McGraw-Hill

3. The Sound Company produces stereos. Their inventory includes four models - the Budget, the Economy, the Executive, and the President models. The Budget needs 50 transistors, 30 capacitors, 7 connectors, and 3 dials. The Economy model needs 65 transistors, 50 capacitors, 9 connectors, and 4 dials. The Executive model needs 85 transistors, 42 capacitors, 10 connectors, and 6 dials. The President model needs 85 transistors, 42 capacitors, 10 connectors, and 12 dials. The daily manufacturing goal in a normal quarter is 10 Budget, 12 Economy, 11 Executive, and 7 President stereos.

   a. How many transistors are needed each day? Capacitors? Connectors? Dials?

   b. During August and September, production is increased by 40%. How many Budget, Economy, Executive, and President models are produced daily during these months?

   c. It takes 5 person-hours to produce the Budget model, 7 person-hours to produce the Economy model, 6 person-hours for the Executive model, and 7 person-hours for the President model. Determine the number of employees needed to maintain the normal production schedule, assuming everyone works an average of 7 hours each day. How many employees are needed in August and September?

4. The president of the Lucrative Bank is hoping for a 21% increase in checking accounts, a 35% increase in savings accounts, and a 52% increase in market accounts. The current statistics on the number of accounts at each branch are as follows:

$$\begin{array}{r c}
 & \begin{array}{ccc} \textit{Checking} & \textit{Savings} & \textit{Market} \end{array} \\
\begin{array}{r} \textit{Northgate} \\ \textit{Downtown} \\ \textit{South Square} \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix}
\end{array}$$

What is the goal for each branch in each type of account? (HINT: multiply by a 3×2 matrix with certain nonzero entries on the diagonal and zero entries elsewhere.) What will be the total number of accounts at each branch?

Lesson 3a - Guided Notes - Student Version

*Student Version begins on the next page.

Solving Linear Systems of Equations Using Inverse Matrices

A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling.

How can we represent this problem with a system of equations?

Let $x =$ ____

Let $y =$ ____

Let $z =$ ____

Definitions:

- The ________________________________ of a square $n\ x\ n$ matrix, A, is an $n\ x\ n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$.

- If an $n\ x\ n$ matrix A^{-1} exists such that $AA^{-1} = I$, then A^{-1} is the ____________________________ of A. (Note that not all matrices have __________. For example, no rectangular matrix (e.g., $2\ x\ 3$) has an __________.)

Example: Consider the following system of linear equations (recall this from Algebra II):

$$\begin{gathered} x + 3y = 0 \\ x + y + z = 1 \\ 3x - y - z = 11 \end{gathered}$$

We can solve this system by representing it using matrices.

We will name the _________________ matrix $A = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$, the **variable vector** $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

and the **column vector** $B = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$. So, our **matrix equation** (also referred to as a linear system of equations) representing the system can be written as $AX = B$:

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

> **Note:** Division is not an operation that is defined for matrices. The analogous operation, however, is multiplying by the **inverse** of a matrix. Just as we divide in order to "reverse" the operation of multiplication between real numbers to return the number 1 (the multiplicative identity in real numbers), we multiply

matrices by their inverses to "reverse" the operation of multiplication between matrices, returning the identity matrix, I.

So, in order to solve the equation $AX = B$ for the matrix X, we will need to do the following, as long as A^{-1} exists:

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

So, back to our problem:

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

We use out calculator to find the inverse of the coefficient matrix, which is

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

The solution to our system, then, is $x =$ _______, $y =$ _______ and $z =$ ________.

Recall: A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling. How much money will each location receive in grants?

Rewrite your system of equations from earlier in this lesson:

We can represent this system using the following linear system:

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Using our calculators to find the inverse of the coefficient matrix $A = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$

we have $A^{-1} \approx \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$. Since the equation $AX = B$ can be solved by $X = A^1 B$, we find

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Therefore, ___________________ goes to the East location, ____________________ goes to the West location, and __________________ goes to the South location.

Exercises

For each of the following problems, identify your variables and write a system of equations to represent the problem. Then use matrices to solve the system.

2. The Frodo Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat is $42 and $30 per acre, respectively. Mr. Frodo has $18,600 available for cultivating these crops. If he wants to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant? (Adapted from *Finite Mathematics,* Tan p. 93 #51)[5]

[5] Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

2. The Coffee Cart sells a blend made with two different coffees, one costing \$2.50 per pound, and the other costing \$3.00 per pound. If the blended coffee sells for \$2.80 per pound, how much of each coffee is used to obtain the blend? (Assume that the weight of the coffee blend is 100 pounds.) (Adapted from *Finite Mathematics,* Tan p. 93 #53)

3. The Maple Movie Theater has a seating capacity of 900 and charges \$2 for children, \$3 for students, and \$4 for adults. At a screening with full attendance last week, there were half as many adults as children and students combined. The receipts totaled \$2800. How many adults attended the show? (Adapted from *Finite Mathematics,* Tan p. 97 #60)

4. The Toolies have a total of \$100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12% per year, while bonds pay 8% per year, and the money market account pays 4% per year. They have decided that the amount invested in stocks should be equal to the difference between the amount invested in bonds and 3 times the amount invested in the money market account. How should the Toolies allocate their resources if they require an annual income of \$10,000 from their investments? (Adapted from *Finite Mathematics,* Tan p. 106 #36)

Lesson 3b - Guided Notes - Student Version

*Student Version begins on the next page.

Solving Linear Systems of Equations Using Gaussian Elimination

A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
Public Service	20%	30%	40%
Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling.

How can we represent this problem with a system of equations?

Let $x =$

Let $y =$

Let $z =$

We therefore have the following system of equations:

Example: Consider the following system of linear equations (recall this from Algebra II):

$$\begin{aligned} x + 3y &= 0 \\ x + y + z &= 1 \\ 3x - y - z &= 11 \end{aligned}$$

We can solve this system by representing it using matrices.

We will name the __________________ matrix $A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$, the **variable vector** $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the **column vector** $B = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$. So, our **matrix equation** (also referred to as a linear system of equations) representing the system can be written as $AX = B$:

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

One way to solve this system is to use an approach known as

____, or **row reduction.**

Gaussian Elimination

You may recall from your prior mathematics work that there are three possible conclusions we can make about the solution to a system of equations.

Case 1: There exists one unique solution.
Case 2: There is no solution.
Case 3: There is an infinite number of solutions.

<u>Case 1</u>: There exists one unique solution.

Recall our example from above:

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix}$$

To begin, we write the associated ___________________________________**,** which is written in the following form:

To apply the method on a matrix, we use ____ to

modify the matrix. Our goal is to end up with the ______________________________, which is an

$n\ x\ n$ matrix with all 1's in the main diagonal and zeros elsewhere: $I = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, on the left side of the augmented matrix.

Our solution to the system of equations will be the resulting matrix on the right side of the augmented matrix. This is because the resulting augmented matrix would represent a system of equations in which each variable could be solved for (if a solution exists).

Elementary Row Operations:

There are three operations that can be applied to modify the matrix and still preserve the solution to the system of equations.

- Exchanging two rows (which represents the switching the listing order of two equations in the system)
- Multiplying a row by a nonzero scalar (which represents multiplying both sides of one of the equations by a nonzero scalar)
- Adding a multiple of one row to another (which represents does not affect the solution, since both equations are in the system)

For our example…

$$\begin{aligned} x + 3y &= 0 && R_1 \\ x + y + z &= 1 && R_2 \\ 3x - y - z &= 11 && R_3 \end{aligned}$$

System of equations	Row operation	Augmented matrix

Back to our opening problem! A business is sponsoring grants for three different projects: scholarships for employees, public service projects, and remodeling of its storefronts. Each of the store locations in Mathtown made requests for funds with the relative amounts requested by each location distributed as shown in the following table:

	Location		
Project	**East**	**West**	**South**
Scholarships	50%	30%	40%
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Remodeling	30%	40%	20%

The corporate office has decided to grant $100,000 for the projects, and they decided to distribute it with 43% to scholarships, 28% to public service and 29% to remodeling. How much money will each location receive in grants?

Rewrite your system of equations from earlier in this lesson:

We can represent this system using the following linear systems of equations:

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$$

The augmented matrix for this system is:

$$\left[\begin{array}{ccc|c} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{array}\right]$$

Using elementary row operations, we find that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \approx \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

So, ____________________ goes to the East location, __________________ goes to the West location,

and ____________________ goes to the South location.

<u>Case 2</u>: There is no solution.

Consider the system of equations:

$$\begin{aligned} 2x - y + z &= 1 \\ 3x + 2y - 4z &= 4 \\ -6x + 3y - 3z &= 2 \end{aligned}$$

Augmented matrix: $$\left[\begin{array}{ccc|c} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{array}\right]$$

Using row operation $R_3 + 3R_1 \rightarrow R_3$, we get

$$\left[\begin{array}{ccc|c} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{array}\right]$$

We note that the third row in the augmented matrix is a false statement, so there is no solution to this system.

<u>Case 3</u>: There is an infinite number of solutions.

Consider the system of equations:

$$\begin{aligned} x - y + 2z &= -3 \\ 4x + 4y - 2z &= 1 \\ -2x + 2y - 4z &= 6 \end{aligned}$$

Augmented matrix:

Using row operations $R_2 - 4R_1 \to R_2$ and $R_3 + 2R_1 \to R_3$, we get

This represents a system that leaves us with 2 equations and 3 unknowns. So, we are unable to solve for one variable without expressing it in terms of another. This gives us an infinite number of solutions.

Exercises

For each of the following problems, identify your variables and write a system of equations to represent the problem. Then use Gaussian elimination to solve the system.

1. The Frodo Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat is $42 and $30 per acre, respectively. Mr. Frodo has $18,600 available for cultivating these crops. If he wants to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant? (Adapted from *Finite Mathematics,* Tan p. 93 #51[6])

2. The Coffee Cart sells a blend made with two different coffees, one costing $2.50 per pound, and the other costing $3.00 per pound. If the blended coffee sells for $2.80 per pound, how much of each coffee is used to obtain the blend? (Assume that the weight of the coffee blend is 100 pounds.) (Adapted from *Finite Mathematics,* Tan p. 93 #53)

[6] Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

3. The Maple Movie Theater has a seating capacity of 900 and charges \$2 for children, \$3 for students, and \$4 for adults. At a screening with full attendance last week, there were half as many adults as children and students combined. The receipts totaled \$2800. How many adults attended the show? (Adapted from *Finite Mathematics,* Tan p. 97 #60)

4. The Toolies have a total of \$100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12% per year, while bonds pay 8% per year, and the money market account pays 4% per year. They have decided that the amount invested in stocks should be equal to the difference between the amount invested in bonds and 3 times the amount invested in the money market account. How should the Toolies allocate their resources if they require an annual income of \$10,000 from their investments? (Adapted from *Finite Mathematics,* Tan p. 106 #36)

Lesson 4 - Guided Notes – Excel - Student Version

*Student Version begins on the next page.

Introduction to Colley's Method

Name: ______________________________

Date: ________________ Period: _______

Given a list of items:

- **Ranking:** ____
- **Rating:** ____

Examples of Rankings/Ratings:

- Sports: ____
- Schools: ____
- Search results: ____
- Social networks: ____

Key Challenges:

- **Objectivity:** ____
- **Transparency:** ____
- **Robustness:** ____

  ____

  ____

  ____

Win/Loss Records:

Can we use just the win/loss records to rank teams?

What are some challenges to considering only the win/loss records?

Considerations for win/loss records:

- How could you account for strength of schedule? What if teams try to play all easy-to-beat teams to earn a higher win/loss record?

- Should we take the margin of victory into account? What if the game is a close game? A blowout?
- Should there be correction for home/away games or other factors?

Colley's Method of Ranking:

Colley's Method of Ranking began as a slight modification to the general ranking based on win percentage. This method has its advantages because it does not rank based on just the win percentage, therefore the teams cannot build their schedules to play easy-to-beat teams and rack up their win percentage to rank higher. This method encourages the teams to play more difficult-to-beat teams, because if they beat those higher ranked teams, then they will earn more points to their own ranking. Colley's method also considers the win/loss record and the total number of games but uses this information differently. The first step is to construct an n x n matrix *C*, which we call Colley's matrix, and an n x 1 vector *b*. The next step is to solve the linear system of equations *Cr=b* to obtain Colley's ratings *r*. Finally, we use the ratings vector to determine the rankings (higher values of *r*, means higher ranking). (Source: Who's #1 The Science of Rating and Ranking)

Variables and their Meanings

$\boldsymbol{N}$: ______________________ $\boldsymbol{w_i}$: ______________________

$\boldsymbol{l_i}$: ______________________ $\boldsymbol{t_i}$: ______________________

$\boldsymbol{r_i}$: ______________________ $\boldsymbol{n_{ij}}$: ______________________

- **Matrix System:**

$$(2 + t_i)r_i - \sum_{j=1}^{N}(n_{ij}r_j) = 1 + \frac{w_i - l_i}{2}$$

To Solve: ***Cr=b***

$$C = \begin{bmatrix} 2+t_1 & -n_{12} & -n_{13} & \cdots & -n_{1N} \\ -n_{21} & 2+t_2 & -n_{23} & \ddots & -n_{2N} \\ -n_{31} & -n_{32} & 2+t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -n_{N1} & \dots & \dots & \cdots & 2+t_N \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad b = \begin{bmatrix} 1 + \frac{w_1 - l_1}{2} \\ 1 + \frac{w_2 - l_2}{2} \\ \vdots \\ 1 + \frac{w_N - l_N}{2} \end{bmatrix}$$

Examples:

7) **College Football Records**

	Duke	**Miami**	**UNC**	**UVA**	**VT**	**Record**
Duke		7-52	21-24	7-38	0-45	0-4
Miami	52-7		34-16	25-17	27-7	4-0
UNC	24-21	16-34		7-5	3-30	2-2
UVA	38-7	17-25	5-7		14-52	1-3
VT	45-0	2-27	30-3	52-14		3-1

$\boldsymbol{t_i} =$ __________ $\quad\quad$ $\boldsymbol{2 + t_i} =$ ____________

$\boldsymbol{n_{ij}} =$ ______________ $\quad\quad$ $\boldsymbol{b} =$ _______________

$$\boldsymbol{C} = \begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ \\ \\ \\ \\ \end{bmatrix} \quad \boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} \quad \\ \\ \\ \\ \\ \end{bmatrix}$$

In Excel: To Calculate C-Inverse: =MINVERSE(*array*) and To Calculate r: =MMULT(*C-Inverse array, r array*)

	C						b	
	6	-1	-1	-1	-1		-1	
	-1	6	-1	-1	-1		3	
	-1	-1	6	-1	-1		1	
	-1	-1	-1	6	-1		0	
	-1	-1	-1	-1	6		2	
							r	
C-inverse	0.2142857	0.071428571	0.071428571	0.071428571	0.071428571		0.21	Duke
	0.0714286	0.214285714	0.071428571	0.071428571	0.071428571		0.79	Miami
	0.0714286	0.071428571	0.214285714	0.071428571	0.071428571		0.50	Unc
	0.0714286	0.071428571	0.071428571	0.214285714	0.071428571		0.36	Uva
	0.0714286	0.071428571	0.071428571	0.071428571	0.214285714		0.64	Vt

8) ***YOU TRY!*** **Movie Ratings**

	Fargo	**Shrek**	**Milk**	**Jaws**
User 1	5	4	3	-
User 2	5	5	3	1
User 3	-	-	-	5
User 4	-	-	2	-
User 5	4	-	-	3
User 6	1	-	-	4

*A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for user 1, Fargo beats Shrek because a 5 is higher than a 4. You should compare all movies in this manner. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

	w_i	l_i	Ties	t_i	$t_i + 2$
Fargo					
Shrek					
Milk					
Jaws					

$$C = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \quad b = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}$$

In Excel:

	C					b	
	9	-2	-2	-3		3	
	-2	7	-2	-1		2	
	-2	-2	7	-1		-0.5	
	-3	-1	-1	7		-0.5	
						r	
C-inverse	0.191860465	0.098837209	0.098837209	0.110465116		0.669	Fargo
	0.098837209	0.2125323	0.101421189	0.087209302		0.627	Shrek
	0.098837209	0.101421189	0.2125323	0.087209302		0.349	Milk
	0.110465116	0.087209302	0.087209302	0.215116279		0.355	Jaws

Lesson 4 - Guided Notes – Gaussian Elimination - Student Version

*Student Version begins on the next page.

Introduction to Colley's Method

Name: ______________________________

Date: ________________ Period: _______

Given a list of items:

- **Ranking:** ____
- **Rating:** ____

Examples of Rankings/Ratings:

- Sports: ____
- Schools: ____
- Search results: ____
- Social networks: ____

Key Challenges:

- **Objectivity:** ____
- **Transparency:** ____
- **Robustness:** ____

  ____

  ____

  ____

Win/Loss Records:

Can we use just the win/loss records to rank teams?

What are some challenges to considering only the win/loss records?

Considerations for win/loss records:

- How could you account for strength of schedule? What if teams try to play all easy-to-beat teams to earn a higher win/loss record?

- Should we take the margin of victory into account? What if the game is a close game? A blowout?
- Should there be correction for home/away games or other factors?

Colley's Method of Ranking:

Colley's Method of Ranking began as a slight modification to the general ranking based on win percentage. This method has its advantages because it does not rank based on just the win percentage, therefore the teams cannot build their schedules to play easy-to-beat teams and rack up their win percentage to rank higher. This method encourages the teams to play more difficult-to-beat teams, because if they beat those higher ranked teams, then they will earn more points to their own ranking. Colley's method also considers the win/loss record and the total number of games but uses this information differently. The first step is to construct an n x n matrix *C*, which we call Colley's matrix, and an n x 1 vector *b*. The next step is to solve the linear system of equations *Cr=b* to obtain Colley's ratings *r*. Finally, we use the ratings vector to determine the rankings (higher values of *r*, means higher ranking). (Source: Who's #1 The Science of Rating and Ranking)

Variables and their Meanings

$\boldsymbol{N}$: ____________________	$\boldsymbol{w_i}$: ____________________
$\boldsymbol{l_i}$: ____________________	$\boldsymbol{t_i}$: ____________________
$\boldsymbol{r_i}$: ____________________	$\boldsymbol{n_{ij}}$: ____________________

- **Matrix System:**

$$\boldsymbol{(2+t_i)r_i - \sum_{j=1}^{N}(n_{ij}r_j) = 1 + \frac{w_i - l_i}{2}}$$

To Solve: ***Cr=b***

$$\boldsymbol{C = \begin{bmatrix} 2+t_1 & -n_{12} & -n_{13} & \cdots & -n_{1N} \\ -n_{21} & 2+t_2 & -n_{23} & \ddots & -n_{2N} \\ -n_{31} & -n_{32} & 2+t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -n_{N1} & \ldots & \ldots & \cdots & 2+t_N \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad b = \begin{bmatrix} 1 + \frac{w_1 - l_1}{2} \\ 1 + \frac{w_2 - l_2}{2} \\ \vdots \\ 1 + \frac{w_N - l_N}{2} \end{bmatrix}}$$

Examples:

9) **College Football Records**

	Duke	Miami	UNC	UVA	VT	Record
Duke		7-52	21-24	7-38	0-45	0-4
Miami	52-7		34-16	25-17	27-7	4-0
UNC	24-21	16-34		7-5	3-30	2-2
UVA	38-7	17-25	5-7		14-52	1-3
VT	45-0	2-27	30-3	52-14		3-1

$\boldsymbol{t_i} =$ _________ $\quad$ $\boldsymbol{2 + t_i} =$ ___________

$\boldsymbol{n_{ij}} =$ ____________ $\quad$ $\boldsymbol{b} =$ _____________

$$\boldsymbol{C} = \begin{bmatrix} \quad \end{bmatrix} \qquad \boldsymbol{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} \quad \end{bmatrix}$$

10) ***YOU TRY!* Movie Ratings**

	Fargo	Shrek	Milk	Jaws
User 1	5	4	3	-
User 2	5	5	3	1
User 3	-	-	-	5
User 4	-	-	2	-
User 5	4	-	-	3
User 6	1	-	-	4

*A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for user 1, Fargo beats Shrek because a 5 is higher than a 4. You should compare all movies in this manner. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

	$\boldsymbol{w_i}$	$\boldsymbol{l_i}$	Ties	$\boldsymbol{t_i}$	$\boldsymbol{t_i + 2}$
Fargo					
Shrek					
Milk					
Jaws					

$$C = \begin{bmatrix} \quad \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \qquad b = \begin{bmatrix} \quad \end{bmatrix}$$

Lesson 4 - Guided Notes – TI84 - Student Version

*Student Version begins on the next page.

Introduction to Colley's Method

Name: ______________________________

Date: ________________ Period: _______

Given a list of items:

- **Ranking:** ____
- **Rating:** ____

Examples of Rankings/Ratings:

- Sports: ____
- Schools: ____
- Search results: ____
- Social networks: ____

Key Challenges:

- **Objectivity:** ____
- **Transparency:** ____
- **Robustness:** ____

  ____

  ____

  ____

Win/Loss Records:

Can we use just the win/loss records to rank teams?

What are some challenges to considering only the win/loss records?

Considerations for win/loss records:

- How could you account for strength of schedule? What if teams try to play all easy-to-beat teams to earn a higher win/loss record?

- Should we take the margin of victory into account? What if the game is a close game? A blowout?
- Should there be correction for home/away games or other factors?

Colley's Method of Ranking:

Colley's Method of Ranking began as a slight modification to the general ranking based on win percentage. This method has its advantages because it does not rank based on just the win percentage, therefore the teams cannot build their schedules to play easy-to-beat teams and rack up their win percentage to rank higher. This method encourages the teams to play more difficult-to-beat teams, because if they beat those higher ranked teams, then they will earn more points to their own ranking. Colley's method also considers the win/loss record and the total number of games but uses this information differently. The first step is to construct an n x n matrix *C*, which we call Colley's matrix, and an n x 1 vector *b*. The next step is to solve the linear system of equations *Cr=b* to obtain Colley's ratings *r*. Finally, we use the ratings vector to determine the rankings (higher values of *r*, means higher ranking). (Source: Who's #1 The Science of Rating and Ranking)

Variables and their Meanings

$\boldsymbol{N}$: ______________________ $\boldsymbol{w_i}$: ______________________

$\boldsymbol{l_i}$: ______________________ $\boldsymbol{t_i}$: ______________________

$\boldsymbol{r_i}$: ______________________ $\boldsymbol{n_{ij}}$: ______________________

- **Matrix System:**

$$(2+t_i)r_i - \sum_{j=1}^{N}(n_{ij}r_j) = 1 + \frac{w_i - l_i}{2}$$

To Solve: ***Cr=b***

$$C = \begin{bmatrix} 2+t_1 & -n_{12} & -n_{13} & \cdots & -n_{1N} \\ -n_{21} & 2+t_2 & -n_{23} & \ddots & -n_{2N} \\ -n_{31} & -n_{32} & 2+t_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -n_{N1} & \dots & \dots & \cdots & 2+t_N \end{bmatrix} \quad r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{bmatrix} \quad b = \begin{bmatrix} 1+\frac{w_1-l_1}{2} \\ 1+\frac{w_2-l_2}{2} \\ \vdots \\ 1+\frac{w_N-l_N}{2} \end{bmatrix}$$

Examples:

11) **College Football Records**

	Duke	**Miami**	**UNC**	**UVA**	**VT**	**Record**
Duke		7-52	21-24	7-38	0-45	0-4
Miami	52-7		34-16	25-17	27-7	4-0
UNC	24-21	16-34		7-5	3-30	2-2
UVA	38-7	17-25	5-7		14-52	1-3
VT	45-0	2-27	30-3	52-14		3-1

$\boldsymbol{t_i} =$ ______________ $\quad$ $\boldsymbol{2 + t_i} =$ ______________

$\boldsymbol{n_{ij}} =$ __________________ $\quad$ $\boldsymbol{b} =$ __________________

$$\boldsymbol{C} = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \quad \boldsymbol{r} = \begin{bmatrix} \boldsymbol{r_1} \\ \boldsymbol{r_2} \\ \boldsymbol{r_3} \\ \boldsymbol{r_4} \\ \boldsymbol{r_5} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

Write the Augmented Matrix:

$$A = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix}$$

Solve for the *"r"* matrix using your TI-84 Calculator:

$$r = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

12) ***YOU TRY!*** **Movie Ratings**

	Fargo	**Shrek**	**Milk**	**Jaws**
User 1	5	4	3	-
User 2	5	5	3	1
User 3	-	-	-	5
User 4	-	-	2	-
User 5	4	-	-	3
User 6	1	-	-	4

*A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for user 1, Fargo beats Shrek because a 5 is higher than a 4. You should compare all movies in this manner. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

	$\boldsymbol{w_i}$	$\boldsymbol{l_i}$	**Ties**	$\boldsymbol{t_i}$	$\boldsymbol{t_i + 2}$
Fargo					
Shrek					
Milk					
Jaws					

$$C = \begin{bmatrix} \quad \end{bmatrix} \qquad r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \qquad b = \begin{bmatrix} \quad \end{bmatrix}$$

Write the Augmented Matrix:

$$A = \begin{bmatrix} \quad \end{bmatrix}$$

Solve for the *"r"* matrix using your TI-84 Calculator:

$$r = \begin{bmatrix} \quad \end{bmatrix}$$

Lesson 4 - Colley's Method Problem Set - Student Version

*Student Version begins on the next page.

Colley's Method Problem Set

At the Movies

Five friends rate five different movies on a scale of 1 to 5. They do not know each other's ratings, and some of them have not seen all of the movies. A movie "wins" if it has a higher rating than the other movie it is "competing" against, i.e. for Madison, Avengers: Endgame beats Toy Story 4, since she rated the former a 4 and the latter a 3. If a movie does not have a rating, then it is not competing in that "round". If there is a tie, then it does not count as a win or a loss.

Movie Title/ Rating	**LOTR: Return of the King**	**Star Wars**	**Toy Story 4**	**Harry Potter and the Sorcerer's Stone**	**Avengers: Endgame**
Madison	5	3	3	2	4
Kelia	4	4	--	4	2
Raffi	2	--	3	1	5
Rachel	5	2	4	2	--
Owen	3	5	4	--	--

4. Complete the following table given the ratings above.

$\boldsymbol{i}$	**Movie $\boldsymbol{i}$**	**# Wins**	**# Losses**	**# Ties**	**# of Comparisons**	$\boldsymbol{b_i = 1 + \frac{w_i - l_i}{2}}$
1	LOTR: Return of the King					
2	Star Wars					
3	Toy Story 4					
4	Harry Potter and the Sorcerer's Stone					
5	Avengers: Endgame					

5. Write the Colley Matrix in the matrix equation and the vector on the right ("$\boldsymbol{b}$" vector) that are associated with the information above.

$$\boldsymbol{C} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

6. Solve for the ratings using technology, and convert to the Colley ranking.

$$\boldsymbol{r} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

$\boldsymbol{i}$	**Movie $\boldsymbol{i}$**	**Colley Rank**
1	LOTR: Return of the King	
2	Star Wars	
3	Toy Story 4	
4	Harry Potter	
5	Avengers: Endgame	

Colley's Method NCAA Division Basketball Problem[7]

The following is data from the games played in the America East conference from January 2, 2013, to January 10, 2013 in the 2013 NCAA Men's Division 1 Basketball. (This data can be found on the ESPN website.)

The teams in the conference are as follows:

i	Team i	Abbreviation
1	Stony Brook	STON
2	Vermont	UVM
3	Boston University	BU
4	Hartford	HART
5	Albany	ALBY
6	Maine	ME
7	Univ. Maryland, Bal. County	UMBC
8	New Hampshire	UNH
9	Binghampton	BING

The following is a record of their games and results (W/L) from January 2, 2013, to January 10, 2013:

Date	**Teams**	**Winner**
Jan 02, 2013	BING vs HART	HART
Jan 02, 2013	UVM vs UNH	UVM
Jan 02, 2013	BU vs ME	ME
Jan 02, 2013	ALBY vs UMBC	ALBY
Jan 05, 2013	STON vs UNH	STON
Jan 05, 2013	UVM vs ALBY	UVM
Jan 05, 2013	BU vs HART	HART
Jan 05, 2013	ME vs UMBC	ME
Jan 07, 2013	BING vs ALBY	ALBY
Jan 08, 2013	UVM vs BU	BU

[7] Source: https://www3.nd.edu/~apilking/Math10170/Information/Lectures%202015/Topic8Colley.pdf

Jan 09, 2013	BING vs STON	STON
Jan 09, 2013	ME vs HART	HART
Jan 09, 2013	UMBC vs UNH	UMBC

2. Complete the following table given the information above.

i	Team i	Abbrev	# Wins	# Losses	# Ties	# of Comparisons	$b_i = 1 + \frac{w_i - l_i}{2}$
1	Stony Brook	STON					
2	Vermont	UVM					
3	Boston University	BU					
4	Hartford	HART					
5	Albany	ALBY					
6	Maine	ME					
7	Univ. Maryland, Bal. County	UMBC					
8	New Hampshire	UNH					
9	Binghampton	BING					

3. Write the Colley Matrix in the matrix equation and the vector on the right ("$\boldsymbol{b}$" vector) that are associated with the information above.

$$\boldsymbol{C} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

4. Solve for the ratings using technology, and convert to the Colley rankings.

$$\boldsymbol{r} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

i	Team i	Abbreviation	Colley Rank
1	Stony Brook	STON	
2	Vermont	UVM	
3	Boston University	BU	
4	Hartford	HART	
5	Albany	ALBY	
6	Maine	ME	
7	Univ. Maryland, Bal. County	UMBC	
8	New Hampshire	UNH	
9	Binghampton	BING	

Lesson 5 – Rock, Paper, Scissors Activity – Student Version

*Student Version begins on the next page.

Rock Paper Scissors

  

Day 1:

Step 1: Form groups of 4-5 students.

Step 2: Play “Rock, Paper, Scissors” with each person in the group playing every other person in the group. (Best ⅔ to determine the winner)

Day 2:

Step 3: Record the Wins/Losses in the Google Form. https://forms.gle/QLVAYYafNE6ZmtzRA

Step 4: Teacher will convert data to a Spreadsheet or organize it for calculations by hand. (Teacher Discretion)

Step 5: Use the Colley’s Method to determine who is the Rock Paper Scissors Champion.

Lesson 6 – Colley's Method Final Project – Student Handout

*Student Version begins on the next page.

Final Project for Ranking with Colley's Method



Day 1 (End of Lesson 5):

Step 1: Choose a topic you are interested in ranking.

Step 2: Either collect data or find data on this topic. For example if you are interested in the NFL you may want to use statistics that are already available on their site. However, if you want to rank music you may want to choose 5 songs and have others rate them and perform Colley's method on those 5 songs. Your choice!

Day 2 (Lesson 6):

Step 3: Organize your data in a table.

Step 4: Perform the steps you learned in the previous lesson using the Colley's Method for Ranking. (This can be done by hand or with Excel depending on your teacher's preference.)

Step 5: Using your results conclude who/what is the best and defend this assertion with the data.

Step 6: Display your results in a way that your classmates can easily understand.

Step 7: Consider any drawbacks or limitations this method had on your data. Are there flaws with this ranking system? If so, what would you suggest as an alternative?