Image Processing

Authors:
Tamar Avineri, North Carolina School of Science and Mathematics, Durham NC
Emily Berkeley, Panther Creek High School, Cary, NC
Ashley Miller, North Rowan High School, Spencer, NC

Advisors: Eric Hallman, Rachel Minster, and Arvind K. Saibaba (asaibab@ncsu.edu)
North Carolina State University

Abstract: In this all-inclusive module students will learn how to use matrices and vectors to manipulate and edit images. The students will be provided with their own image editing activity where they will create their own classroom "Instagram feed." There is also an extension activity that incorporates some basic Python programming into this module.

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Implementation Notes

- **Length of module:**
  - In total, this unit is designed to take approximately 3.5 days of 90-minute lessons, or 7 days of 45-minute lessons, including the final project. There is also an extension project that is designed to take approximately 85 minutes.
  - Each of the lessons is accompanied by an estimate of the length of time it is designed to take in class. If the estimate is longer than you can devote in class, feel free to select portions for students to complete outside of class.
  - There is an extension project that includes coding in Python. This is a lesson that is intended for Discrete Math for Computer Science but would be a great tool if you have the extra time to implement the lesson in the Math 4 or Pre-Calculus class.

- **Relevant courses:** This module is designed to be self-contained, as the first 2 lessons provide foundational knowledge in the linear algebra skills that students will need for the subsequent lessons. The materials are appropriate for any NC Math 4, Pre-Calculus, or Discrete Mathematics for Computer Science courses. This could also serve as an interesting study following the AP exam for students in AP Calculus AB or BC.

- **Mathematical practices/student learning outcomes:** In addition to the standards for mathematical practices, this module addresses several standards covered in NC Math 4, Pre-Calculus, and Discrete Mathematics for Computer Science.
  - *Mathematical practices*:
    - Make sense of problems and persevere in solving them.
    - Reason abstractly and quantitatively.
    - Construct viable arguments and critique the reasoning of others.
    - Model with mathematics.
    - Use appropriate tools strategically.
    - Attend to precision.
    - Look for and make use of structure.
    - Look for and express regularity in repeated reasoning.
    - Use strategies and procedures flexibly.
    - Reflect on mistakes and misconceptions.
  - ***NC Math 4:*** NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; NC.M4.N.2.2 Execute procedures of addition, subtraction, and scalar multiplication on vectors.
  - ***Precalculus:*** PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; PC.N.2.5 Execute the multiplication algorithm with matrices.
  - ***Discrete Mathematics for Computer Science*:** DCS.N.1.1 Implement procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; DCS.N.1.2 Implement procedures of addition, subtraction, and scalar multiplication

on vectors; DCS.N.1.3 Implement procedures to find the inverse of a matrix; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation; DCS.N.2.4 Use inverse matrices to solve a system of equations with technology.

- **Assessments:**
  - Feel free to select portions of the guided notes to serve as out-of-class activities.
  - Any problem set contained within guided notes could be given as homework assignments.
  - As an alternative to the final project in this module, you could choose to give students a standard test or quiz on the skills that have been learned.

- **Online delivery suggestions:**
  - For asynchronous online delivery, create instructional videos to take students through the guided notes.
  - For synchronous online delivery, display the guided notes on your screen and take students through the activities while you annotate on your screen (or writing on paper and using a document camera).
  - Share all prepared documents through a learning management system so that students would have access to them at home
  - For the project, you can have the students create the matrices using Microsoft word (or if they have access to a tablet and a note taking app, they can create the matrices on there), they can use this website to create the pictures, and then they can screen snip the pictures out and paste them into the final document. They can then submit the project fully through an online submission.

- **List of accompanying documents:**
  - Lesson 4: Python or Jupyter Coding Template
    - To download Jupyter click HERE

- **Student Versions:** Please note that the student versions are located at the end of this document in Appendix A.

Lesson 1: Introduction to Matrices and Matrix Operations

Lesson Plan

<table>
<tr>
<td colspan="2">Standards
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices

PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar;

DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices.</td>
<td colspan="2">Topic/Day: Introduction to Matrices, Matrix Addition/Subtraction, and Scalar Multiplication
Content Objective: Elementary Matrix Operations
Vocabulary: matrix; row; column; dimension; square; transpose

(~60 minutes)</td>
</tr>
<tr>
<td></td>
<td>Time</td>
<td>Student Does</td>
<td>Teacher Does</td>
</tr>
</table>

Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~15 min	Students read the opening problem (e.g., Textbook Problem or other context of interest) from a handout and/or projected on a screen. In groups of 2-3, students talk briefly about how they would answer the question from the teacher. (~5 minutes) The teacher brings back students to share out with the class. (~3 minutes) Students are provided guided notes to document new terms (e.g., matrix, dimension, row, column, etc.) They will complete the notes through the discussion conducted by the teacher. (~5 minutes)	Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, "How might we organize this information in a way that allows us to answer questions about the university's inventory?" After students discuss, the teacher solicits students' responses. If students do not suggest a matrix, the teacher will introduce the name and ask students if they are familiar with the term. If not, the teacher will define it through one of the matrices used to organize the information in the problem.
Explore I Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~15 min	Students complete the second matrix from the problem in their groups. (~5 min) Students work in groups of 2-3 to answer teacher's question. (~5 minutes)	Teacher circulates the room to observe/monitor students' work. Teacher then poses question: "How could we use these matrices to determine the total inventory of books at the university?" Once students have some time to answer question, teacher returns to full class discussion to ask how we could define matrix addition.

Explain I Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~5 min	Students offer their ideas on how to define matrix addition (and subtraction). Students engage in class discussion on teachers' questions.	Teacher conducts discussion on matrix addition and subtraction. Teacher poses questions: "Is matrix addition commutative? Is it associative? Is matrix subtraction commutative? Is it associative? Why/why not?"
Explore II Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~5 min	Students work in groups of 2-3 to answer teacher's question. (~5 minutes)	Teacher then poses question: "How could we use these matrices to determine the inventory of books at the university if the librarian would like to double the inventory?" Once students have some time to answer question, teacher returns to full class discussion to ask how we could define scalar multiplication..
Explain II Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~5 min	Students offer their ideas on how to define scalar multiplication.	Teacher conducts discussion on scalar multiplication.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~15 min	Students complete class problem set in groups of 2-3 to apply their new knowledge.	Teacher circulates the room and observes/monitors students' work.

Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	N/A	Students will complete guided notes and a problem set for practice. Students turn in their solutions to the last problem in the problem set as an exit ticket (e.g., Stereo Problem)	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking.
Evaluate **Summative-like Assessment** Questions similar to final assessment	N/A	N/A	N/A

Guided Notes – Teacher Version

Matrix Addition, Subtraction and Scalar Multiplication

A university is taking inventory of the books they carry at their two biggest bookstores.
The East Campus bookstore carries the following books:

Hardcover: Textbooks-5280; Fiction-1680; NonFiction-2320; Reference-1890

Paperback: Textbooks-1930; Fiction-2705; NonFiction-1560; Reference-2130

The West Campus bookstore carries the following books:

Hardcover: Textbooks-7230; Fiction-2450; NonFiction-3100; Reference-1380

Paperback: Textbooks-1740; Fiction-2420; NonFiction-1750; Reference-1170

In order to work with this information, we can represent the inventory of each bookstore using an organized array of numbers known as a *matrix*.

Definitions: A **matrix** is a rectangular table of entries and is used to organize data in a way that can be used to solve problems. The following is a list of terms used to describe matrices:

- A matrix's **size (or dimension)** is written by listing the number of rows "by" the number of columns.
- The values in a matrix, A, are referred to as **entries** or **elements**. The entry in the "m^{th}" row and "n^{th}" column is written as a_{mn}.
- A matrix is **square** if it has the same number of rows as it has columns.
- If a matrix has only one row, then it is a row **vector**. If it has only one column, then the matrix is a column **vector.**
- The **transpose** of a matrix, A, written A^T, switches the rows with the columns of A and the columns with the rows.
- Two matrices are **equal** if they have the same size and the same corresponding entries.

The inventory of the books at the East Campus bookstore can be represented with the following **2 x 4** matrix:

$$E = \begin{array}{c} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} \end{array}$$

Similarly, the West Campus bookstore's inventory can be represented with the following matrix:

$$W = \begin{array}{c} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 7230 & 2450 & 3100 & 1380 \\ 1740 & 2420 & 1750 & 1170 \end{bmatrix} \end{array}$$

Adding and Subtracting Matrices

In order to add or subtract matrices, they must first be of the same **size**. The result of the addition or subtraction is a matrix of the same size as the matrices themselves, and the entries are obtained by adding or subtracting the elements in corresponding positions.

In our campus bookstores example, we can find the total inventory between the two bookstores as follows:

$$E + W = \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} + \begin{bmatrix} 7230 & 2450 & 3100 & 1380 \\ 1740 & 2420 & 1750 & 1170 \end{bmatrix}$$

$$= \begin{array}{c} \\ Hardback \\ Paperback \end{array} \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 12510 & 4130 & 5420 & 3270 \\ 3670 & 5125 & 3310 & 3300 \end{bmatrix} \end{array}$$

Question: *Is matrix addition commutative* $(e.g., A + B = B + A)$? *Why or why not?*
Matrix addition is commutative. This is because the operation is based in the addition of real numbers, as the entries of each matrix are added to their corresponding entries in the other matrix/matrices. Since addition of real numbers is commutative, so is matrix addition.

Question: *Is matrix subtraction commutative* $(e.g., A - B = B - A)$? *Why or why not?*
Matrix subtraction is not commutative. This is because the operation is based in the subtraction of real numbers, as the entries of each matrix are subtracted from their corresponding entries in the other matrix/matrices. Since subtraction of real numbers is not commutative, neither is matrix subtraction.

Question: *Is matrix addition associative* $(e.g., (A + B) + C = A + (B + C))$? *Why or why not?*
Matrix addition is associative. This is because the operation is based in the addition of real numbers, as the entries of each matrix are added to their corresponding entries in the other matrix/matrices. Since addition of real numbers is associative, so is matrix addition.

Question: *Is matrix subtraction associative* $(e.g., (A - B) - C = A - (B - C))$? *Why or why not?*
Matrix subtraction is not associative. This is because the operation is based in the subtraction of real numbers, as the entries of each matrix are subtracted from their corresponding entries in the other matrix/matrices. Since subtraction of real numbers is not associative, neither is matrix subtraction.

Scalar Multiplication

Multiplying a matrix by a constant (or *scalar*) is as simple as multiplying each entry by that number! Suppose the bookstore manager in East Campus wants to double his inventory. He can find the number of books of each type that he would need by simply multiplying the matrix *E* by the scalar (or constant) 2. The result is as follows:

$$2E = 2 * \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 5280 & 1680 & 2320 & 1890 \\ 1930 & 2705 & 1560 & 2130 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{bmatrix} 2(5280) & 2(1680) & 2(2320) & 2(1890) \\ 2(1930) & 2(2705) & 2(1560) & 2(2130) \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{r} Hardback \\ Paperback \end{array} & \begin{bmatrix} 10560 & 3360 & 4640 & 3780 \\ 3860 & 5410 & 3120 & 4260 \end{bmatrix} \end{array}$$

Exercises: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed:

a. $A + B$: This operation cannot be performed, since matrices A and B are of different dimensions.

b. $B - A$: This operation also cannot be performed, as A and B have different dimensions.

c. $$A - C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{-6} & \mathbf{22} \\ \mathbf{0} & \mathbf{-8} & \mathbf{12} \\ \mathbf{-11} & \mathbf{8} & \mathbf{7} \end{bmatrix}$$

d. $$C - A = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{6} & \mathbf{-22} \\ \mathbf{0} & \mathbf{8} & \mathbf{-12} \\ \mathbf{11} & \mathbf{-8} & \mathbf{-7} \end{bmatrix}$$

e. $5B = 5 * \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{40} & \mathbf{-30} \end{bmatrix}$

f. $$-A + 4C = -\begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} + 4 * \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 0 & -1 \\ -2 & 4 & -3 \\ 6 & -1 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 24 & -84 \\ 8 & 16 & -36 \\ 20 & -28 & 4 \end{bmatrix} = \begin{bmatrix} \mathbf{-1} & \mathbf{24} & \mathbf{-85} \\ \mathbf{6} & \mathbf{20} & \mathbf{-39} \\ \mathbf{26} & \mathbf{-29} & \mathbf{-4} \end{bmatrix}$$

g. $B - D$: This operation cannot be performed, since B and D are not of the same size.

h. $$2C - 6A = 2 * \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} - 6 * \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 12 & -42 \\ 4 & 8 & -18 \\ 10 & -14 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 6 \\ 12 & -24 & 18 \\ -36 & 6 & 48 \end{bmatrix} = \begin{bmatrix} \mathbf{-6} & \mathbf{12} & \mathbf{-48} \\ \mathbf{-8} & \mathbf{32} & \mathbf{-36} \\ \mathbf{46} & \mathbf{-20} & \mathbf{-46} \end{bmatrix}$$

i. $B^T + D = \begin{bmatrix} 2 \\ 8 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{7} \\ \mathbf{6} \\ \mathbf{-3} \end{bmatrix}$

Lesson 2: Matrix Multiplication

Lesson Plan

<table>
<tr><td colspan="2">Standards

NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices

PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar;

DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context</td><td colspan="2">Topic/Day: Matrix Multiplication
Content Objective: Elementary Matrix Operations

(~70 minutes)</td></tr>
<tr><td></td><td>Time</td><td>Student Does</td><td>Teacher Does</td></tr>
<tr><td>Warm Up
Elicit/Engage
Build relevance through a problem
Try to find out what your students already know
Get them interested</td><td>~5 min</td><td>Students read the opening problem (e.g., Opera Problem or other context of interest) from a handout and/or projected on a screen.</td><td>Teacher hands out a sheet of paper with the opening problem written on it and/or projects it on the screen. Teacher opens with the question, “How might we organize this information in a way that allows us to answer the question?”</td></tr>
</table>

Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~15 min	In groups of 2-3, students work together to calculate each value of interest by hand (not using any specific method). (~10 minutes) Students work in groups of 2-3 to answer teacher's question. (~10 minutes) Students can break the work up among their group members.	Teacher asks students to calculate each value of interest by hand, showing their work but not using any specific method. The teacher brings students back to share their results and confirm their results with other groups.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~15 min	Students follow along the teachers' explanation on their opening problem. Students share their thoughts on teacher's posed questions.	Teacher conducts lesson on matrix multiplication using the opening problem to demonstrate the operation. Teacher poses questions: "Is matrix multiplication commutative? Is it associative? Why/why not?" Teacher provides examples of why they are/aren't, and students practice the operation with those examples.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~25 min	Students complete class problem set in groups of 2-3 to apply their new knowledge.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate Assessment How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.

Guided Notes – Teacher Version

Matrix Multiplication

The Metropolitan Opera is planning its last cross-country tour. It plans to perform *Carmen* and *La Traviata* in Atlanta in May. The person in charge of logistics wants to make plane reservations for the two troupes. *Carmen* has 2 stars, 25 other adults, 5 children, and 5 staff members. *La Traviata* has 3 stars, 15 other adults, and 4 staff members. There are 3 airlines to choose from. Redwing charges round-trip fares to Atlanta of $630 for first class, $420 for coach, and $250 for youth. Southeastern charges $650 for first class, $350 for coach, and $275 for youth. Air Atlanta charges $700 for first class, $370 for coach, and $150 for youth. Assume stars travel first class, other adults and staff travel coach, and children travel for the youth fare.

Use multiplication and addition to find the total cost for each troupe to travel each of the airlines.

Carmen/Redwing: $2(630) + 30(420) + 5(250) = \$15110$

Carmen/Southeastern: $2(650) + 30(350) + 5(275) = \$13175$

Carmen/Air Atlanta: $2(700) + 30(370) + 5(150) = \$13250$

La Traviata/Redwing: $3(630) + 19(420) + 0(250) = \$9870$

La Traviata/Southeastern: $3(650) + 19(350) + 0(275) = \$8600$

La Traviata/Air Atlanta: $3(700) + 19(370) + 0(150) = \$9130$

It turns out that we can solve problems like these using a matrix operation, specifically **matrix multiplication**!

We first note that matrix multiplication is only defined for matrices of certain sizes. For the product AB of matrices A and B, where A is an $m\ x\ n$ matrix, B must have the same number of rows as A has columns. So, B must have size $\boldsymbol{n}\ x\ p$. The product AB will have size $\boldsymbol{m\ x\ p}$.

Exercises

The following is a set of abstract matrices (without row and column labels):

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad O = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 5 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

List at least 5 orders of pairs of matrices from this set for which the product is defined. State the dimension of each product.

MO: 2x1 **MP: 2x2** **PM: 2x2** **MR: 2x2** **RM: 2x2** **NQ: 3x1**

NU: 3x4 **PO: 2x1** **US: 3x2** **UT: 3x1**

Back to the opera…

Define two matrices that organize the information given:

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{stars} & \textbf{adults} & \textbf{children} \end{array} \\ \textbf{\textit{Carmen}} & \multirow{2}{*}{$\begin{bmatrix} 2 & 30 & 5 \\ 3 & 19 & 0 \end{bmatrix}$} \\ \textbf{\textit{La Traviata}} & \end{array}$$

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} & \begin{bmatrix} 630 & 650 & 700 \\ 420 & 350 & 370 \\ 250 & 275 & 150 \end{bmatrix} \end{array}$$

We can multiply these two matrices to obtain the same answers we obtained above, all in one matrix!

$$\begin{array}{r c} & \begin{array}{ccc} \textbf{stars} & \textbf{adults} & \textbf{children} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 2 & 30 & 5 \\ 3 & 19 & 0 \end{bmatrix} \end{array} \cdot \begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} & \begin{bmatrix} 630 & 650 & 700 \\ 420 & 350 & 370 \\ 250 & 275 & 150 \end{bmatrix} \end{array}$$

$$= \begin{array}{r c} & \begin{array}{ccc} \textbf{Red} & \textbf{South} & \textbf{Air} \end{array} \\ \begin{array}{r} \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} & \begin{bmatrix} 15110 & 13175 & 13250 \\ 9870 & 8600 & 9130 \end{bmatrix} \end{array}$$

Carmen/Redwing: $15110

Carmen/Southeastern: $13175

Carmen/Air Atlanta: $13250

La Traviata/Redwing: $9870

La Traviata/Southeastern: $8600

La Traviata/Air Atlanta: $9130

Exercises

1. The K.L. Mutton Company has investments in three states - North Carolina, North Dakota, and New Mexico. Its deposits in each state are divided among bonds, mortgages, and consumer loans. The amount of money (in millions of dollars) invested in each category on June 1 is displayed in the table below.

	NC	ND	NM
Bonds	13	25	22
Mort.	6	9	4
Loans	29	17	13

The current yields on these investments are 7.5% for bonds, 11.25% for mortgages, and 6% for consumer loans. Use matrix multiplication to find the total earnings for each state.

Total earnings for each state (in millions of dollars):

$$\begin{array}{ccc} \boldsymbol{Bonds} & \boldsymbol{Mort.} & \boldsymbol{Loans} \\ \end{array} \atop [\mathbf{1.075} \quad \mathbf{1.1125} \quad \mathbf{1.06}] \quad \begin{array}{c|ccc} & \boldsymbol{NC} & \boldsymbol{ND} & \boldsymbol{NM} \\ \boldsymbol{Bonds} & \mathbf{13} & \mathbf{25} & \mathbf{22} \\ \boldsymbol{Mort.} & \mathbf{6} & \mathbf{9} & \mathbf{4} \\ \boldsymbol{Loans} & \mathbf{29} & \mathbf{17} & \mathbf{13} \end{array} = \begin{array}{ccc} \boldsymbol{NC} & \boldsymbol{ND} & \boldsymbol{NM} \end{array} \atop [\mathbf{3.39} \quad \mathbf{3.9075} \quad \mathbf{2.88}]$$

2. Several years ago, Ms. Allen invested in growth stocks, which she hoped would increase in value over time. She bought 100 shares of stock A, 200 shares of stock B, and 150 shares of stock C. At the end of each year she records the value of each stock. The table below shows the price per share (in dollars) of stocks A, B, and C at the end of the years 1984, 1985, and 1986.

	1984	1985	1986
Stock A	68.00	72.00	75.00
Stock B	55.00	60.00	67.50
Stock C	82.50	84.00	87.00

Calculate the total value of Ms. Allen's stocks at the end of each year.

Total value of the stocks (in dollars) at the end of each year:

$$\begin{array}{ccc} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{C} \end{array} \atop [\mathbf{100} \quad \mathbf{200} \quad \mathbf{150}] \quad \begin{array}{c|ccc} & \mathbf{1984} & \mathbf{1985} & \mathbf{1986} \\ \boldsymbol{A} & \mathbf{68} & \mathbf{72} & \mathbf{75} \\ \boldsymbol{B} & \mathbf{55} & \mathbf{60} & \mathbf{67.5} \\ \boldsymbol{C} & \mathbf{82.5} & \mathbf{84} & \mathbf{87} \end{array} = \begin{array}{ccc} \mathbf{1984} & \mathbf{1985} & \mathbf{1986} \end{array} \atop [\mathbf{30,175} \quad \mathbf{31,800} \quad \mathbf{34,050}]$$

3. The Sound Company produces stereos. Their inventory includes four models - the Budget, the Economy, the Executive, and the President models. The Budget needs 50 transistors, 30 capacitors, 7 connectors, and 3 dials. The Economy model needs 65 transistors, 50 capacitors, 9 connectors, and 4 dials. The Executive model needs 85 transistors, 42 capacitors, 10 connectors, and 6 dials. The President model needs 85 transistors, 42 capacitors, 10 connectors, and 12 dials. The daily manufacturing goal in a normal quarter is 10 Budget, 12 Economy, 11 Executive, and 7 President stereos.

a. How many transistors are needed each day? Capacitors? Connectors? Dials?

b. During August and September, production is increased by 40%. How many Budget, Economy, Executive, and President models are produced daily during these months?

c. It takes 5 person-hours to produce the Budget model, 7 person-hours to produce the Economy model, 6 person-hours for the Executive model, and 7 person-hours for the President model. Determine the number of employees needed to maintain the normal production schedule, assuming everyone works an average of 7 hours each day. How many employees are needed in August and September?

Define the matrices for the inventory parts $(\boldsymbol{I})$ and the daily manufacturing goal $(\boldsymbol{N})$ as

$$\boldsymbol{I} = \begin{array}{c} \\ \boldsymbol{B} \\ \boldsymbol{Ec} \\ \boldsymbol{Ex} \\ \boldsymbol{P} \end{array} \begin{array}{c} \begin{array}{cccc} \boldsymbol{t} & \boldsymbol{ca} & \boldsymbol{co} & \boldsymbol{d} \end{array} \\ \begin{bmatrix} \mathbf{50} & \mathbf{30} & \mathbf{7} & \mathbf{3} \\ \mathbf{65} & \mathbf{50} & \mathbf{9} & \mathbf{4} \\ \mathbf{85} & \mathbf{42} & \mathbf{10} & \mathbf{6} \\ \mathbf{85} & \mathbf{42} & \mathbf{10} & \mathbf{12} \end{bmatrix} \end{array} \quad \boldsymbol{and} \quad \boldsymbol{N} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{B} & \boldsymbol{Ec} & \boldsymbol{Ex} & \boldsymbol{P} \end{array} \\ \begin{bmatrix} \mathbf{10} & \mathbf{12} & \mathbf{11} & \mathbf{7} \end{bmatrix} \end{array}$$

a. The answers are the results of the matrix multiplication

$$\boldsymbol{NI} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{t} & \boldsymbol{ca} & \boldsymbol{co} & \boldsymbol{d} \end{array} \\ \begin{bmatrix} \mathbf{2810} & \mathbf{1656} & \mathbf{358} & \mathbf{228} \end{bmatrix} \end{array}$$

b. The new daily manufacturing goals are given by

$$\mathbf{1.4}\boldsymbol{N} = \begin{array}{c} \begin{array}{cccc} \boldsymbol{B} & \boldsymbol{Ec} & \boldsymbol{Ex} & \boldsymbol{P} \end{array} \\ \begin{bmatrix} \mathbf{14} & \mathbf{16.8} & \mathbf{15.4} & \mathbf{9.8} \end{bmatrix} \end{array}$$

Which should be rounded to integer quantities

c. Define a matrix $\boldsymbol{H}$ for hours of labor as

$$\boldsymbol{H} = \begin{array}{c} \\ \boldsymbol{B} \\ \boldsymbol{Ec} \\ \boldsymbol{Ex} \\ \boldsymbol{P} \end{array} \begin{array}{c} \boldsymbol{Hrs.} \\ \begin{bmatrix} \mathbf{5} \\ \mathbf{7} \\ \mathbf{6} \\ \mathbf{7} \end{bmatrix} \end{array}$$

The number of labor hours needed per week is given by

$$\boldsymbol{NH} = \mathbf{249}$$

With 7-hour workdays, the number of employees needed is $\frac{249}{7} = 35.6$, which implies that 36 employees are needed to maintain full production. For August and September, we want $\frac{1.4NH}{7} = \frac{348.6}{7}$, which rounds to 50.

4. The president of the Lucrative Bank is hoping for a 21% increase in checking accounts, a 35% increase in savings accounts, and a 52% increase in market accounts. The current statistics on the number of accounts at each branch are as follows:

$$\begin{array}{rc} & \begin{array}{ccc} \textit{Checking} & \textit{Savings} & \textit{Market} \end{array} \\ \begin{array}{r} \textit{Northgate} \\ \textit{Downtown} \\ \textit{South Square} \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array}$$

What is the goal for each branch in each type of account? (HINT: multiply by a 3×2 matrix with certain nonzero entries on the diagonal and zero entries elsewhere.) What will be the total number of accounts at each branch?

The goal for each branch in each type of account is given by:

$$\begin{array}{rc} & \begin{array}{ccc} c & s & m \end{array} \\ \begin{array}{r} N \\ S \\ D \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array} \cdot \begin{array}{rc} & \begin{array}{ccc} c & s & m \end{array} \\ \begin{array}{r} c \\ s \\ m \end{array} & \begin{bmatrix} 1.21 & 0 & 0 \\ 0 & 1.35 & 0 \\ 0 & 0 & 1.52 \end{bmatrix} \end{array}$$

$$= \begin{array}{rc} & \begin{array}{ccc} c & s & m \end{array} \\ \begin{array}{r} N \\ S \\ D \end{array} & \begin{bmatrix} 48447 & 13682 & 778.24 \\ 18430 & 11814 & 159.6 \\ 30991 & 16452 & 147.44 \end{bmatrix} \end{array}$$

Right-multiplying this result by the matrix $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields the following total number of accounts at each branch: $\begin{array}{rc} & \mathit{Total} \\ \begin{array}{r} N \\ D \\ S \end{array} & \begin{bmatrix} 62907.68 \\ 30402.96 \\ 47590.41 \end{bmatrix} \end{array}$.

Note: this answer can also be obtained by just adding up the entries in each row of the previous matrix.

Lesson 3: Introduction to Image Processing

Lesson Plan

Standard **DCS.N.1.1 Implement procedures of addition, subtraction, multiplication and scalar multiplication on matrices.**	**Topic/Day: Introduction to Image Processing** **Content Objective:** **Language Objective:** **Vocabulary:** **Materials Needed:** **(~90 minutes)**

	Time	Student Does	Teacher Does
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~10 min	Students work on nonogram activity in groups of 2-3. (Activity site)	Teacher opens with a color-by-number nanogram-style image displayed to the class and prompts students to complete the diagram. (Teacher provides the rules/process of completing a nanogram.) Teacher opens discussion with students about what pixels are and asks students if they know how to interpret the numbers. Ultimately, teacher concludes that 0 represents black and 255 represents white.
Explore I Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~10 min	Students work in groups of 2-3 through guided notes provided by the teacher. The teacher will work through the first example with the students and then give them time to work the rest. The guided notes would include both problems on drawing images based on matrices and vice versa.	Teacher demonstrates how to draw an image based on a matrix with the students (displaying the first matrix on the students' guided notes on the screen for students to follow.)

			Teacher circulates the room monitoring students' progress and addressing questions along the way. The teacher brings students back to share and confirm their results.
Explain Ia Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~10 min	Students follow along the teachers' explanation. Students then work together in groups of 2-3 on practice problems based on the teacher's lesson.	Teacher conducts lesson on interpreting what the sum, difference, scalar multiplication and combinations of operations of matrices would represent in the context of images. Teacher brings back class to discuss and confirm results.
Explain Ib Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~10 minutes	Students follow along the teachers' explanation.	Teacher conducts lesson on "bits"
Explore II Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~10 min	Students work in groups of 2-3 through guided notes provided by the teacher. The teacher will work through the first example with the students and then give them time to work the rest.	Teacher demonstrates how to convert matrices into vectors (displaying the first matrix on the students' guided notes on the screen for students to follow.) Teacher circulates the room monitoring students' progress and addressing questions along the way. The teacher brings students back to share and confirm their results.

Explain II Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~15 min	Students engage in discussion as a class, facilitated by the teacher. They share their results up at the board and present them to the class.	Teacher debriefs the results from the Explore session and conducts class discussion on results.
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	~20 min	Students apply their new understanding to a problem set involving specific images.	Teacher will review students' work as they circulate room and monitor progress, engaging students who may be having difficulty in discussion to probe their thinking. Teacher brings class back together to engage in debrief on the problem set.
Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	~10 min	Students work on exit ticket problem and turn it in.	Teacher poses exit ticket problem for students to turn in.
Evaluate **Summative-like Assessment** Questions similar to final assessment	N/A	Instagram Project (separate lesson)	N/A

Guided Notes – Teacher Version

Introduction to Image Processing Name: ______________________________

Part I: Image representation and processing Date: ________________ Period: _______

In this lesson, we will discuss representations of images and methods to manipulate those images.

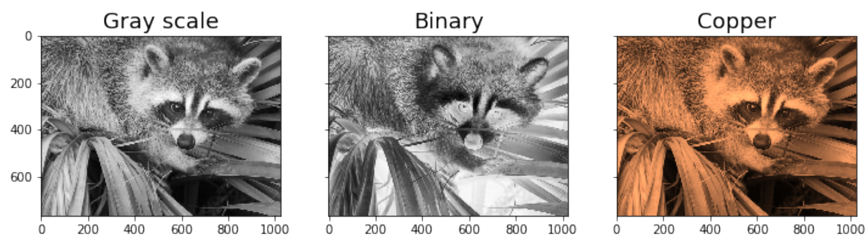


Figure: Image of a raccoon in grayscale, binary and copper coloring

Matrix representations:

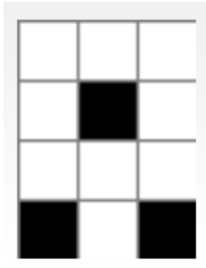
- A grayscale image is a 2-dimensional array of numbers. An 8-bit image has entries between **0** and **255**.
- The value **255** represents a white color, and the value **0** represents a black color.
- Lower numbers translate to **darker** pixels, while higher numbers translate to **lighter** pixels.
- For an image that has $m \ x \ n$ pixels (i.e., "picture elements"), we represent that image using a matrix of size $\boldsymbol{m \ x \ n}$. The entries of the matrix indicate the pixel value of the corresponding part of the image.

Example: This table represents an image that has $4 \ x \ 5$ pixels.

Example: This $4 \ x \ 4$ image of the raccoon is represented by the matrix on the right below:

```
[[114 130 145 147]
 [ 83 104 123 130]
 [ 68  88 109 116]
 [ 78  94 109 116]]
```

Example: Write the matrix that represents the following image:

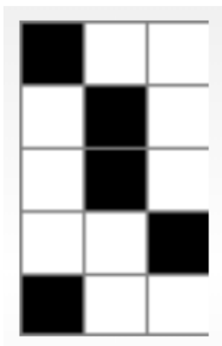


$$\begin{bmatrix} 255 & 255 & 255 \\ 255 & 0 & 255 \\ 255 & 255 & 255 \\ 0 & 255 & 0 \end{bmatrix}$$

Exercises:

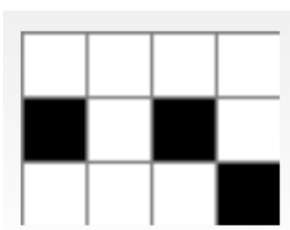
1. Write a matrix that represents each of the following images.

a.



$$\begin{bmatrix} 0 & 255 & 255 \\ 255 & 0 & 255 \\ 255 & 0 & 255 \\ 255 & 255 & 0 \\ 255 & 0 & 255 \end{bmatrix}$$

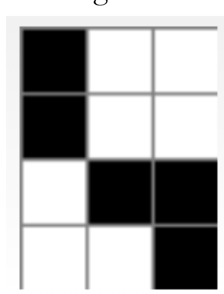
b.



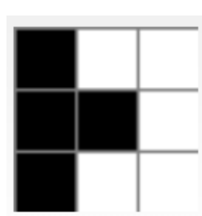
$$\begin{bmatrix} 255 & 255 & 255 & 255 \\ 0 & 255 & 0 & 255 \\ 255 & 255 & 255 & 0 \end{bmatrix}$$

2. Draw a visual representation of the following matrices.

a. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$



b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$



c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$

Color Images:

Color images can be stored in a similar fashion to a grayscale image. Instead of one number (0-255) per pixel, one stores three numbers per pixel – these three numbers denote the "amount" of red, "amount" of green, and "amount" of blue in each pixel. These three numbers can be used to depict a wide range of colors.

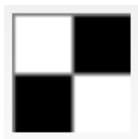
Image transformations:

Once images are represented as matrices, we can describe many transformations of those images using basic matrix operations!

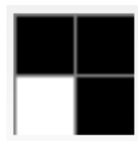
Example: Suppose we have two separate images that can be represented by the matrices $A = \begin{bmatrix} 255 & 0 \\ 0 & 255 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 255 & 0 \end{bmatrix}$.

1. Draw the images that correspond to A and B.

A

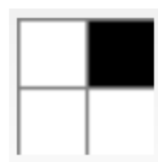


B



2. Find and draw $A + B$.

$A + B = \begin{bmatrix} 255 & 0 \\ 0 & 255 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 255 & 0 \end{bmatrix} = \begin{bmatrix} 255 & 0 \\ 255 & 255 \end{bmatrix}$



3. Compute $C = \frac{1}{5}A$. How would you describe the image represented by C in relation to the image represented by A?

$C = \frac{1}{5}A = \frac{1}{5}\begin{bmatrix} 255 & 0 \\ 0 & 255 \end{bmatrix} = \begin{bmatrix} 51 & 0 \\ 0 & 51 \end{bmatrix}$

The image represented by the matrix C would have pixels that are darker than those in the image represented by matrix A.

Note: When mathematical operations on images result in entries that range outside of the integer values 0 through 255, image visualization tools treat them as 255 if the value exceeds 255, and as 0 if the value is below 0.

Exercises:

1. a. Describe an image whose matrix representation contains entries of all zeros.

The image would be entirely black.

b. Describe an image whose matrix representation contains entries of all 255.

The image would be entirely white.

2. Let A and B be matrices that represent two images of the same size. Describe in words what each of the following means.

   a. $A - B$

This matrix represents the image that is the result of darkening the image represented by A, if B has at least one nonzero entry.

b. $5A + B$

This matrix represents the lightening of the image represented by A, since all of the values in A are multiplied by a positive number, and the entries in B are added to those entries.

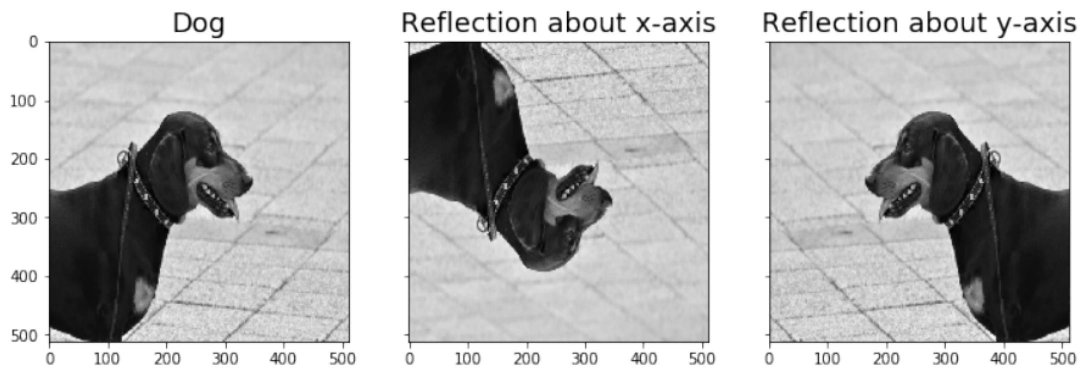
   c. $255 * 1 - A$, where "1" is the matrix of all 1's

This matrix is the result of darkening an all-white image, since the entries in A are all positive numbers (if A has at least one nonzero entry.

Part II: Manipulating images

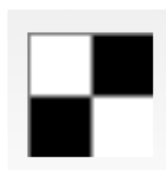
Reflecting/rotating images

Let us consider how we might reflect an image over the *x* or *y*-axis. Here's an example!

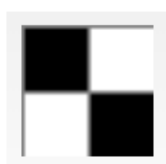


How would we represent these reflections using matrices?

From our earlier work, we know that the image represented by the matrix $X = \begin{bmatrix} 255 & 0 \\ 0 & 255 \end{bmatrix}$ would be:



If we were to reflect this image about the *x*-axis, we would obtain:



The matrix X' that represents this image is $X = \begin{bmatrix} 0 & 255 \\ 255 & 0 \end{bmatrix}$. We can generalize this result!

Exercises:

1. Which matrix represents the image that is the reflection of the matrix $A = \begin{bmatrix} 0 & a & b & 0 \\ c & d & 0 & 0 \\ e & 0 & f & g \\ 0 & 0 & 0 & h \end{bmatrix}$ about the x-axis?

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{h} \\ \boldsymbol{e} & \mathbf{0} & \boldsymbol{f} & \boldsymbol{g} \\ \boldsymbol{c} & \boldsymbol{d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{a} & \boldsymbol{b} & \mathbf{0} \end{bmatrix}$$

2. Write the matrix that corresponds to the reflection of each of the following about the
   a. x −axis
   b. y −axis

a. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$ x −axis: $\begin{bmatrix} 255 & 255 & 0 \\ 255 & 0 & 0 \\ 0 & 255 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

y −axis: $\begin{bmatrix} 255 & 255 & 0 \\ 255 & 255 & 0 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$ x −axis: $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

y −axis: $\begin{bmatrix} 255 & 255 & 0 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$ x −axis: $\begin{bmatrix} 0 & 255 & 0 \\ 255 & 0 & 0 \end{bmatrix}$

y −axis: $\begin{bmatrix} 0 & 0 & 255 \\ 0 & 255 & 0 \end{bmatrix}$

Reflecting images using matrix-matrix multiplication

Consider the matrix $X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ that represents an image[1].

Question: From our work above, we know that the matrix representing the image reflected about the x-axis is $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. What matrix would we need to multiply X by on the **left** in order to obtain the matrix of the image reflected about the $\boldsymbol{x}$**-axis**? Show that your answer gives you the correct result.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Question: From our work above, we know that the matrix representing the image reflected about the y-axis is $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. What matrix would we need to multiply X by on the **right** in order to obtain the matrix of the image reflected about the $\boldsymbol{y}$**-axis**? Show that your answer gives you the correct result.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

The matrix you used above is a form of a **permutation** matrix (here referred to as a reverse identity matrix). This matrix is always **square**, and its size is determined by the size of the matrix it is multiplied to (i.e. in order for the matrix multiplication to be performed).

Exercises:

1. What size does the reverse identity matrix need to be to be used to reflect an image represented by a $5\ x\ 3$ matrix about the x-axis? Why?

The matrix must be a $5\ x\ 5$ matrix, so that it can left-multiply the $5\ x\ 5$ matrix. The reverse identity matrix is square, and it must have the same number of columns as the $5\ x\ 3$ matrix has rows.

[1] Note that these values each represent a shade of color between white and black, where 0 represents black, and 1 represents white. While we have used only values of 0 and 255 thus far, we will use these values in this example to make the reflections clear.

2. What size does the reverse identity matrix need to be to be used to reflect an image represented by a $5 \; x \; 3$ matrix about the y-axis? Why?

The matrix must be a $3 \; x \; 3$ matrix, so that it can right-multiply the $5 \; x \; 3$ matrix. The reverse identity matrix is square, and it must have the same number of rows as the $5 \; x \; 3$ matrix has columns.

3. Use the reverse identity matrix of the appropriate size to reflect the images represented by the following matrices about the x-axis.

a. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix} = \begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix} = \begin{bmatrix} 255 & 255 & 0 \\ 255 & 0 & 0 \\ 0 & 255 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 255 & 0 \\ 255 & 0 & 0 \end{bmatrix}$

4. Use the reverse identity matrix of the appropriate size to reflect the images represented by the following matrices about the y-axis.

a. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 255 & 255 & 0 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 255 & 255 & 0 \\ 255 & 255 & 0 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 255 \\ 0 & 255 & 0 \end{bmatrix}$

Reflecting images using matrix-vector multiplication

We can also manipulate images using matrix-vector multiplication. This method is a bit more powerful than matrix-matrix multiplication, because it allows us with more flexibility in ways you can manipulate those images.

Using this method, we first convert our matrices into vectors (i.e., single column or single row matrices).

- We define $vec(X)$ as the vector conversion of the matrix X. In general, for $X = \begin{bmatrix} \vdots & & \vdots \\ x_1 & \dots & x_n \\ \vdots & & \vdots \end{bmatrix}$, we have $vec(X) = \begin{bmatrix} \vdots \\ x_1 \\ \vdots \\ \dots \\ \vdots \\ x_n \\ \vdots \end{bmatrix}$.

- We define $mat(X)$ as the matrix conversion of the vector Y. In general, for $Y = \begin{bmatrix} \vdots \\ x_1 \\ \vdots \\ \dots \\ \vdots \\ x_n \\ \vdots \end{bmatrix}$, we have $mat(X) = \begin{bmatrix} \vdots & & \vdots \\ x_1 & \dots & x_n \\ \vdots & & \vdots \end{bmatrix}$.

Examples:

1. For $X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $vec(X) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

2. For $X = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $vec(X) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}$.

3. For $X = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 1 \end{bmatrix}$, $mat(X) = \begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix}$.

4. For $X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $mat(X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Exercises:

1. Convert each of the following matrices into vectors.

   a. $A = \begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix}$ $\qquad vec(A) = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 4 \end{bmatrix}$.

   b. $B = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$ $\qquad vec(B) = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{bmatrix}$.

   c. $C = \begin{bmatrix} 8 & 8 & 8 \\ 7 & 7 & 7 \\ 8 & 8 & 8 \end{bmatrix}$ $\qquad vec(C) = \begin{bmatrix} 8 \\ 7 \\ 8 \\ 8 \\ 7 \\ 8 \\ 8 \\ 7 \\ 8 \end{bmatrix}$.

2. Convert each of the following vectors into matrices.

a. $X = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 1 \end{bmatrix}$ $mat(X) = \begin{bmatrix} 3 & 6 \\ 5 & 1 \end{bmatrix}$

b. $Y = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ $mat(Y) = \begin{bmatrix} 8 & 5 & 2 \\ 7 & 4 & 1 \\ 6 & 3 & 0 \end{bmatrix}$

Reflection about the axes using matrix-vector multiplication

Consider the matrix $X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. We know that $vec(X) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Question: From our earlier work, we know that the reflection of the image with matrix X about the x-axis would be $Y = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. So, $vec(Y) = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}$. What matrix would we need to multiply $vec(X)$ by on the **left** in order to obtain $vec(Y)$? Show that your answer gives you the correct result.

$\boldsymbol{y = Ax}$**, where** $\boldsymbol{y}$ **is** $\boldsymbol{vec(Y)}$ **and** $\boldsymbol{x}$ **is** $\boldsymbol{vec(X)}$

Using only 1s and 0s to fill in matrix ***A*****:** $\begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Matrix ***A*** **is a permutation matrix which permutes the entries for** ***x*****.**

Question: From our earlier work, we know that the reflection of the image with matrix X about the y-axis would be $Y = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. So, $vec(Y) = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$. What matrix would we need to multiply $vec(X)$ by on the **right** in order to obtain $vec(Y)$? Show that your answer gives you the correct result.

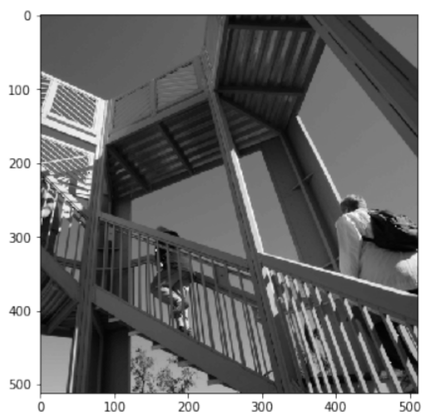
$\boldsymbol{y = Ax}$**, where** $\boldsymbol{y}$ **is** $\boldsymbol{vec(Y)}$ **and** $\boldsymbol{x}$ **is** $\boldsymbol{vec(X)}$

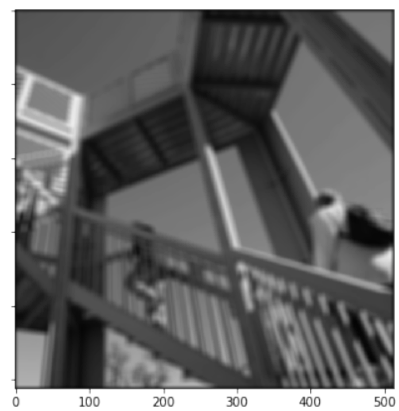
Using only 1s and 0s to fill in matrix ***A*****:** $\begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Matrix ***A*** **is a permutation matrix which permutes the entries for** ***x*****.**

Blurring images

We can also blur images to get an effect we're interested in, and we use linear algebra to do it! The image on the left below is the picture, and the image on the right is the blurred version.





One way to do this is to average each entry's "neighbors" (including the entry itself). The definition of these neighbors can be determined by the user themselves. For example, a neighbor could be any entry in the matrix representation that is directly left, right, above and below the original entry, along with the entry itself.

Example: Consider an image whose matrix representation is $X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$. Suppose we define a "neighbor" as an entry that is directly above, below, next to or diagonal from that entry. For example, the entries that are neighbors of $x_{31} = 3$ in the matrix are $x_{21} = 2$, $x_{22} = 1$, $x_{32} = 1$, and $x_{31} = 3$. So the entry y_{31} in the new matrix would be $\frac{2+1+1+3}{4} = \frac{7}{4}$.

This gives us $Y = \begin{bmatrix} 5/4 & 8/6 & 5/4 \\ 9/6 & 14/9 & 8/6 \\ 7/4 & 10/6 & 5/4 \end{bmatrix}$.

Challenge Question: Find the $9\ x\ 9$ matrix A such that $Y = A * vec(X)$. Show that your choice indeed results in Y.

$$A = \begin{bmatrix} 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 \\ 1/6 & 1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 & 1/6 & 0 \\ 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 & 1/9 \\ 0 & 1/6 & 1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 \end{bmatrix}, vec(X) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

*Note: the denominator of the fractions in matrix *A* are determined by the number of neighbors the element has (example: element x_{31} has 4 neighbors, so in matrix *A* row 3 all has denominators of 4). The numerator will always be 1. Place the appropriate fraction in the location that correlates to where the neighbors of that element are placed in $vec(X)$.

Lesson 4: (Extension) Image Processing using Computer Science

Lesson Plan

Teacher: Subject:

This will be a teacher led activity especially if students are not familiar with Python. If students have previous Python programming experience they could try the coding on their own.	**Topic/Day: Using Python to Manipulate Photos** **Content Objective: Use Matrices to solve real world problems.** **Materials Needed: Python notebook-nothing has to be downloaded but you will need to allow "colab" to run as an extension in Google Chrome to run the program.**

	Time	Student Does	Teacher Does
Warm Up **Elicit/Engage** Build relevance through a problem Try to find out what your students already know Get them interested	~10 min	Write a matrix on paper of the image of the math symbols.	Provide the image either on the board or a copy on paper.
Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~45 min	Explore putting the code in Python to see how the images are created and manipulated. (Code will be provided.)	Provides code and scaffolding.
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~30 min	Student should ask for help on any of the previously taught matrix operations as needed.	Review the algorithms for reflecting, rotating, blurring and combining as needed.

Evaluate **Summative-like Assessment** Questions similar to final assessment	N/A	Final Code will be the assessment	N/A

Lesson 5: Instagram Project

Lesson Plan

<table>
<tr><td colspan="2">Standard
NC.M4.N.2.1 Execute procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; NC.M4.N.2.2 Execute procedures of addition, subtraction, and scalar multiplication on vectors.

PC.N.2.1 Execute the sum and difference algorithms to combine matrices of appropriate dimensions; PC.N.2.2 Execute associative and distributive properties to matrices; PC.N.2.3 Execute commutative property to add matrices; PC.N.2.4 Execute properties of matrices to multiply a matrix by a scalar; PC.N.2.5 Execute the multiplication algorithm with matrices.

DCS.N.1.1 Implement procedures of addition, subtraction, multiplication, and scalar multiplication on matrices; DCS.N.1.2 Implement procedures of addition, subtraction, and scalar multiplication on vectors; DCS.N.1.3 Implement procedures to find the inverse of a matrix; DCS.N.2.1 Organize data into matrices to solve problems; DCS.N.2.2 Interpret solutions found using matrix operations in context; DCS.N.2.3 Represent a system of equations as a matrix equation; DCS.N.2.4 Use inverse matrices to solve a system of equations with technology.</td><td colspan="2">Topic/Day: Instagram Project
Content Objective: Use matrices to manipulate and edit a photo to post on Instagram
Language Objective:
Vocabulary:
Materials Needed: Instagram Project printout, stock images printed, poster board (or other arts and craft supplies for the students)

~85 minutes</td></tr>
<tr><td></td><td>Time</td><td>Student Does</td><td>Teacher Does</td></tr>
<tr><td>Warm Up
Elicit/Engage
Build relevance through a problem
Try to find out what your students already know
Get them interested</td><td>~25 mins</td><td>Students will manipulate the image provided (Lesson 5 – Warm Up) to reflect, blur, and brighten the image</td><td>Provide the students with the worksheet for the warm up and have them write the matrix of the image. Then they would need to manipulate the image and draw the outcome for each manipulation.</td></tr>
</table>

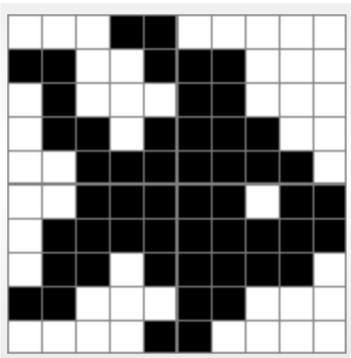
Explore Connectivity to build understanding of concepts Allow for collaboration consider heterogeneous groups Move deliberately from concrete to abstract Apply scaffolding & personalization	~5 mins	Review the Instagram project as a class	Go over the outline and expectations with the students Provide the students with the necessary arts and craft supplies for the project to be completed
Explain Personalize/Differentiate as needed Adjust along teacher/student centered continuum Provide vocabulary Clarify understandings	~45 mins	Complete the Instagram project	Walk around the class listening to the conversations of students and facilitate conversations and answer questions as needed
Extend Apply knowledge to new scenarios Continue to personalize as needed Consider grouping homogeneously	N/A	N/A	N/A
Evaluate **Formative Assessment** How will you know if students understand throughout the lesson?	~10 mins	Share their outcomes with their peers	Facilitate the students sharing their outcomes with their peers You can choose to do whole group presentations or partner groups and have them share with only one group at a time and do a rotation (if you do this, be sure to move around and listen to each groups presentation) Recommended: Hang the outcomes on the wall and create a classroom "Instagram feed" on the wall. You can also have the students give the photos likes/comments and choose a classroom influencer based on the likes/comments that each photo receives.
Evaluate **Summative-like Assessment** Questions similar to final assessment	N/A	N/A	N/A

Warm-Up – Teacher Version

Image Processing | Name: ______________________________

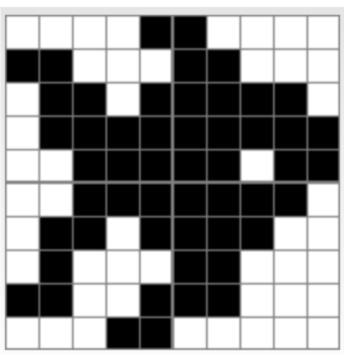
Warm-Up | Date: ___________________Period: ____

Given the image below, write the corresponding matrix:



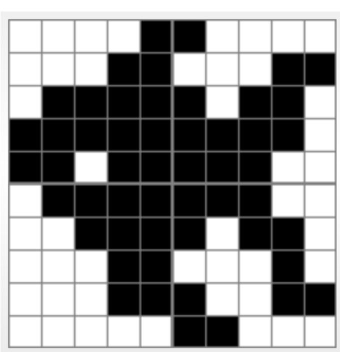
$$A = \begin{bmatrix} 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 255 & 255 \\ 0 & 0 & 255 & 255 & 0 & 0 & 0 & 255 & 255 & 255 \\ 255 & 0 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 \\ 255 & 0 & 0 & 255 & 0 & 0 & 0 & 0 & 255 & 255 \\ 255 & 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 255 \\ 255 & 255 & 0 & 0 & 0 & 0 & 0 & 255 & 0 & 0 \\ 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 255 & 0 & 0 & 255 & 0 & 0 & 0 & 0 & 0 & 255 \\ 0 & 0 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 \\ 255 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 255 \end{bmatrix}$$

Reflect the image over the x-axis, write the new matrix, and draw the new image:



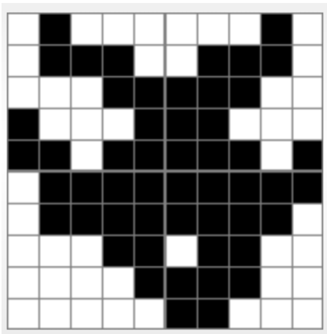
$$A = \begin{bmatrix} 255 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 255 \\ 0 & 0 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 \\ 255 & 0 & 0 & 255 & 0 & 0 & 0 & 0 & 0 & 255 \\ 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 255 & 255 & 0 & 0 & 0 & 0 & 0 & 255 & 0 & 0 \\ 255 & 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 255 \\ 255 & 0 & 0 & 255 & 0 & 0 & 0 & 0 & 255 & 255 \\ 255 & 0 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 \\ 0 & 0 & 255 & 255 & 0 & 0 & 0 & 255 & 255 & 255 \\ 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 255 & 255 \end{bmatrix}$$

Reflect the previous image over the y-axis, write the new matrix, and draw the new image:



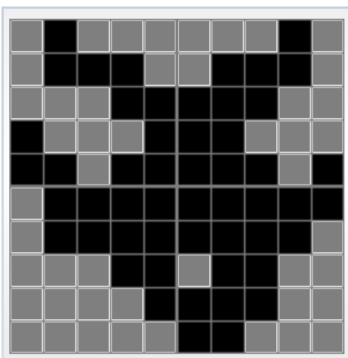
$$A = \begin{bmatrix} 255 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 255 \\ 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 0 & 0 \\ 255 & 0 & 0 & 0 & 0 & 0 & 255 & 0 & 0 & 255 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 255 \\ 0 & 0 & 255 & 0 & 0 & 0 & 0 & 0 & 255 & 255 \\ 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 255 & 255 \\ 255 & 255 & 0 & 0 & 0 & 0 & 255 & 0 & 0 & 255 \\ 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 & 0 & 255 \\ 255 & 255 & 255 & 0 & 0 & 0 & 255 & 255 & 0 & 0 \\ 255 & 255 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 \end{bmatrix}$$

Reflect the previous image over the 45⁰ line, write the new matrix, and draw the new image:



$$A=\begin{bmatrix} 255 & 0 & 255 & 255 & 255 & 255 & 255 & 255 & 0 & 255 \\ 255 & 0 & 0 & 0 & 255 & 255 & 0 & 0 & 0 & 255 \\ 255 & 255 & 255 & 0 & 0 & 0 & 0 & 0 & 255 & 255 \\ 0 & 255 & 255 & 255 & 0 & 0 & 0 & 255 & 255 & 255 \\ 0 & 0 & 255 & 0 & 0 & 0 & 0 & 0 & 255 & 0 \\ 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 255 \\ 255 & 255 & 255 & 0 & 0 & 255 & 0 & 0 & 255 & 255 \\ 255 & 255 & 255 & 255 & 0 & 0 & 0 & 0 & 255 & 255 \\ 255 & 255 & 255 & 255 & 255 & 0 & 0 & 255 & 255 & 255 \end{bmatrix}$$

Change the brightness of the previous image by a scalar of 1/2, write the new matrix, and draw the new image: *Note: you will use the color gray to draw the darkened or lightened image.



$$A=\begin{bmatrix} \frac{255}{2} & 0 & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & 0 & \frac{255}{2} \\ \frac{255}{2} & 0 & 0 & 0 & \frac{255}{2} & \frac{255}{2} & 0 & 0 & 0 & \frac{255}{2} \\ \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & 0 & 0 & 0 & 0 & 0 & \frac{255}{2} & \frac{255}{2} \\ 0 & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & 0 & 0 & 0 & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} \\ 0 & 0 & \frac{255}{2} & 0 & 0 & 0 & 0 & 0 & \frac{255}{2} & 0 \\ \frac{255}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{255}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{255}{2} \\ \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & 0 & 0 & \frac{255}{2} & 0 & 0 & \frac{255}{2} & \frac{255}{2} \\ \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & 0 & 0 & 0 & 0 & \frac{255}{2} & \frac{255}{2} \\ \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} & 0 & 0 & \frac{255}{2} & \frac{255}{2} & \frac{255}{2} \end{bmatrix}$$

- **Did you darken or lighten the image? Why?**
  - Darkened the image because 255 = white, and when the number gets closer to 0, then the shade gets closer to black, so all of the white spaces turned gray.

Blur the original image and write the new matrix (you do not need to draw the new image):

$$
\boldsymbol{A} = \begin{bmatrix}
\frac{255}{2} & 170 & 170 & \frac{255}{2} & 85 & 85 & 170 & \frac{425}{2} & 255 & 255 \\
\frac{255}{2} & 170 & 170 & 170 & \frac{340}{3} & 85 & \frac{425}{3} & \frac{595}{3} & 255 & 255 \\
85 & \frac{340}{3} & \frac{425}{3} & 170 & \frac{340}{3} & \frac{85}{3} & \frac{170}{3} & \frac{425}{3} & \frac{680}{3} & 255 \\
170 & \frac{425}{3} & \frac{340}{3} & \frac{340}{3} & 85 & \frac{85}{3} & \frac{85}{3} & 85 & 170 & \frac{425}{2} \\
\frac{425}{2} & \frac{425}{3} & 85 & \frac{85}{3} & \frac{85}{3} & 0 & \frac{85}{3} & \frac{170}{3} & \frac{340}{3} & \frac{255}{2} \\
\frac{425}{2} & \frac{425}{3} & \frac{170}{3} & 0 & 0 & 0 & \frac{85}{3} & \frac{85}{3} & \frac{170}{3} & \frac{85}{2} \\
170 & \frac{340}{3} & \frac{170}{3} & \frac{85}{3} & \frac{85}{3} & 0 & \frac{85}{3} & \frac{85}{3} & \frac{170}{3} & \frac{85}{2} \\
85 & 85 & 85 & \frac{340}{3} & 85 & \frac{85}{3} & \frac{85}{3} & \frac{170}{3} & \frac{340}{3} & \frac{255}{2} \\
\frac{255}{2} & \frac{425}{3} & 170 & 170 & \frac{340}{3} & \frac{170}{3} & 85 & \frac{425}{3} & \frac{595}{3} & \frac{425}{2} \\
\frac{255}{2} & 170 & \frac{425}{2} & \frac{425}{2} & \frac{255}{2} & 85 & \frac{255}{2} & \frac{425}{2} & 255 & 255
\end{bmatrix}
$$

*Image Source: https://www.woojr.com/printable-picross-grid-puzzles/

Instagram Project

- The project outline is in Appendix A
- There is an example of the Instagram Project in Appendix B

References and Additional Readings

- ❖ Allali, Mohamed. "Linear algebra and image processing." *International Journal of Mathematical Education in Science and Technology* 41.6 (2010): 725-741.

- ❖ Linear algebra and discrete image processing https://www.nibcode.com/en/blog/1135/linear-algebra-and-digital-image-processing-part-I

- ❖ Lecture notes for image processing and computer vision https://staff.fnwi.uva.nl/r.vandenboomgaard/IPCV20172018/LectureNotes/index.html

- ❖ Bartkovich, K. G., Goebel, J. A., Graves, J. L., Teague, D. J., Barrett, G. B., Compton, H. L., ... & Whitehead, K. (2000). *Contemporary Precalculus through Applications*. New York: Glencoe/McGraw-Hill

❖ 9:59

- ❖ Tan, S. (2002). *Finite Mathematics for the Managerial, Life, and Social Sciences* (7th ed.). Boston: Brooks Cole.

Appendices

Appendix A: Lesson Materials – Student Versions

Lesson 1: Guided Notes – Student Version

- Student version begins on next page

Matrix Addition, Subtraction and Scalar Multiplication

A university is taking inventory of the books they carry at their two biggest bookstores.
The East Campus bookstore carries the following books:

Hardcover: Textbooks-5280; Fiction-1680; NonFiction-2320; Reference-1890

Paperback: Textbooks-1930; Fiction-2705; NonFiction-1560; Reference-2130

The West Campus bookstore carries the following books:

Hardcover: Textbooks-7230; Fiction-2450; NonFiction-3100; Reference-1380

Paperback: Textbooks-1740; Fiction-2420; NonFiction-1750; Reference-1170

In order to work with this information, we can represent the inventory of each bookstore using an organized array of numbers known as a *matrix*.

Definitions: A __________ is a rectangular table of entries and is used to organize data in a way that can be used to solve problems. The following is a list of terms used to describe matrices:

- A matrix's __________________________ is written by listing the number of rows "by" the number of columns.
- The values in a matrix, A, are referred to as _____________ or _____________. The entry in the "m^{th}" row and "n^{th}" column is written as a_{mn}.
- A matrix is _____________ if it has the same number of rows as it has columns.
- If a matrix has only one row, then it is a row ____________. If it has only one column, then the matrix is a column _____________.
- The _________________ of a matrix, A, written A^T, switches the rows with the columns of A and the columns with the rows.
- Two matrices are _____________ if they have the same size and the same corresponding entries.

The inventory of the books at the East Campus bookstore can be represented with the following **2 x 4** matrix:

$$E = \begin{array}{cc} & \begin{matrix} T & F & N & R \end{matrix} \\ \begin{matrix} Hardback \\ Paperback \end{matrix} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Similarly, the West Campus bookstore's inventory can be represented with the following matrix:

$$W = \begin{array}{cc} & \begin{matrix} T & F & N & R \end{matrix} \\ \begin{matrix} Hardback \\ Paperback \end{matrix} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Adding and Subtracting Matrices

In order to add or subtract matrices, they must first be of the same ________________________. The result of the addition or subtraction is a matrix of the same size as the matrices themselves, and the entries are obtained by adding or subtracting the elements in corresponding positions.

In our campus bookstores example, we can find the total inventory between the two bookstores as follows:

$$E + W = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} + \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}$$

$$= \begin{array}{cc} & \begin{matrix} T & F & N & R \end{matrix} \\ \begin{matrix} Hardback \\ Paperback \end{matrix} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Question: *Is matrix addition commutative* $(e.g., A + B = B + A)$*? Why or why not?*

Question: *Is matrix subtraction commutative* $(e.g., A - B = B - A)$*? Why or why not?*

Question: *Is matrix addition associative* $(e.g., (A + B) + C = A + (B + C))$*? Why or why not?*

Question: *Is matrix subtraction associative* $(e.g., (A - B) - C = A - (B - C))$*? Why or why not?*

Scalar Multiplication

Multiplying a matrix by a constant (or *scalar*) is as simple as multiplying each entry by that number! Suppose the bookstore manager in East Campus wants to double his inventory. He can find the number of books of each type that he would need by simply multiplying the matrix E by the scalar (or constant) 2. The result is as follows:

$$2E = 2 * \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} = \begin{array}{cc} & \begin{array}{cccc} T & F & N & R \end{array} \\ \begin{array}{c} Hardback \\ Paperback \end{array} & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

Exercises: Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 3 \\ -6 & 1 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 8 & -6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 6 & -21 \\ 2 & 4 & -9 \\ 5 & -7 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

Find each of the following, or explain why the operation cannot be performed:

b. $A + B$

b. $B - A$

$c.$ $A - C$

d. $C - A$

e. $5B$

f. $-A + 4C$

g. $B - D$

h. $2C - 6A$

i. $B^T + D$

Lesson 2: Guided Notes – Student Version

- Student version begins on next page

Matrix Multiplication

The Metropolitan Opera is planning its last cross-country tour. It plans to perform *Carmen* and *La Traviata* in Atlanta in May. The person in charge of logistics wants to make plane reservations for the two troupes. *Carmen* has 2 stars, 25 other adults, 5 children, and 5 staff members. *La Traviata* has 3 stars, 15 other adults, and 4 staff members. There are 3 airlines to choose from. Redwing charges round-trip fares to Atlanta of $630 for first class, $420 for coach, and $250 for youth. Southeastern charges $650 for first class, $350 for coach, and $275 for youth. Air Atlanta charges $700 for first class, $370 for coach, and $150 for youth. Assume stars travel first class, other adults and staff travel coach, and children travel for the youth fare.

Use multiplication and addition to find the total cost for each troupe to travel each of the airlines.

It turns out that we can solve problems like these using a matrix operation, specifically **matrix multiplication**!

We first note that matrix multiplication is only defined for matrices of certain sizes. For the product AB of matrices A and B, where A is an $m \ x \ n$ matrix, B must have the same number of rows as A has columns. So, B must have size _____ $x \ p$. The product AB will have size _____________.

Exercises

The following is a set of abstract matrices (without row and column labels):

$$M = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \quad O = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} \quad U = \begin{bmatrix} 4 & 2 & 6 & -1 \\ 5 & 3 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix}$$

List at least 5 orders of pairs of matrices from this set for which the product is defined. State the dimension of each product.

Back to the opera…

Define two matrices that organize the information given:

$$\begin{array}{r} \\ \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} \begin{array}{c} \textbf{stars} \quad \textbf{adults} \quad \textbf{children} \\ \left[\begin{array}{ccc} & & \\ & & \end{array} \right] \end{array} \qquad \begin{array}{r} \\ \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} \begin{array}{c} \textbf{Red} \quad \textbf{South} \quad \textbf{Air} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array}$$

We can multiply these two matrices to obtain the same answers we obtained above, all in one matrix!

$$\begin{array}{r} \\ \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} \begin{array}{c} \textbf{stars} \quad \textbf{adults} \quad \textbf{children} \\ \left[\begin{array}{ccc} & & \\ & & \end{array} \right] \end{array} \cdot \begin{array}{r} \\ \textbf{stars} \\ \textbf{adults} \\ \textbf{children} \end{array} \begin{array}{c} \textbf{Red} \quad \textbf{South} \quad \textbf{Air} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array}$$

$$= \begin{array}{r} \\ \textbf{\textit{Carmen}} \\ \textbf{\textit{La Traviata}} \end{array} \begin{array}{c} \textbf{Red} \qquad\quad \textbf{South} \qquad\quad \textbf{Air} \\ \left[\begin{array}{ccc} & & \\ & & \end{array} \right] \end{array}$$

Carmen/Redwing:

Carmen/Southeastern:

Carmen/Air Atlanta:

La Traviata/Redwing:

La Traviata/Southeastern:

La Traviata/Air Atlanta:

Exercises[2]

3. The K.L. Mutton Company has investments in three states - North Carolina, North Dakota, and New Mexico. Its deposits in each state are divided among bonds, mortgages, and consumer loans. The amount of money (in millions of dollars) invested in each category on June 1 is displayed in the table below.

	NC	ND	NM
Bonds	13	25	22
Mort.	6	9	4
Loans	29	17	13

The current yields on these investments are 7.5% for bonds, 11.25% for mortgages, and 6% for consumer loans. Use matrix multiplication to find the total earnings for each state.

4. Several years ago, Ms. Allen invested in growth stocks, which she hoped would increase in value over time. She bought 100 shares of stock A, 200 shares of stock B, and 150 shares of stock C. At the end of each year she records the value of each stock. The table below shows the price per share (in dollars) of stocks A, B, and C at the end of the years 1984, 1985, and 1986.

	1984	1985	1986
Stock A	68.00	72.00	75.00
Stock B	55.00	60.00	67.50
Stock C	82.50	84.00	87.00

Calculate the total value of Ms. Allen's stocks at the end of each year.

[2] Adapted from Bartkovich, K. G., Goebel, J. A., Graves, J. L., Teague, D. J., Barrett, G. B., Compton, H. L., ... & Whitehead, K. (2000). *Contemporary Precalculus through Applications*. New York: Glencoe/McGraw-Hill

3. The Sound Company produces stereos. Their inventory includes four models - the Budget, the Economy, the Executive, and the President models. The Budget needs 50 transistors, 30 capacitors, 7 connectors, and 3 dials. The Economy model needs 65 transistors, 50 capacitors, 9 connectors, and 4 dials. The Executive model needs 85 transistors, 42 capacitors, 10 connectors, and 6 dials. The President model needs 85 transistors, 42 capacitors, 10 connectors, and 12 dials. The daily manufacturing goal in a normal quarter is 10 Budget, 12 Economy, 11 Executive, and 7 President stereos.

   a. How many transistors are needed each day? Capacitors? Connectors? Dials?

   b. During August and September, production is increased by 40%. How many Budget, Economy, Executive, and President models are produced daily during these months?

   c. It takes 5 person-hours to produce the Budget model, 7 person-hours to produce the Economy model, 6 person-hours for the Executive model, and 7 person-hours for the President model. Determine the number of employees needed to maintain the normal production schedule, assuming everyone works an average of 7 hours each day. How many employees are needed in August and September?

4. The president of the Lucrative Bank is hoping for a 21% increase in checking accounts, a 35% increase in savings accounts, and a 52% increase in market accounts. The current statistics on the number of accounts at each branch are as follows:

$$\begin{array}{r c} & \begin{array}{ccc} \textit{Checking} & \textit{Savings} & \textit{Market} \end{array} \\ \begin{array}{r} \textit{Northgate} \\ \textit{Downtown} \\ \textit{South Square} \end{array} & \begin{bmatrix} 40039 & 10135 & 512 \\ 15231 & 8751 & 105 \\ 25612 & 12187 & 97 \end{bmatrix} \end{array}$$

What is the goal for each branch in each type of account? (HINT: multiply by a 3×2 matrix with certain nonzero entries on the diagonal and zero entries elsewhere.) What will be the total number of accounts at each branch?

Lesson 3: Guided Notes – Student Version

- Student version begins on next page

Introduction to Image Processing Name: ______________________________

Part I: Image representation and processing Date: ________________ Period: _______

In this lesson, we will discuss representations of images and methods to manipulate those images.

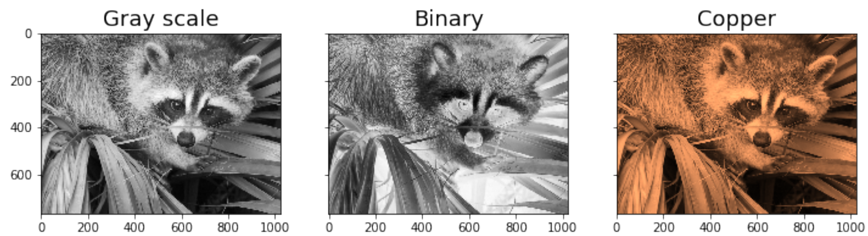


Figure: Image of a raccoon in grayscale, binary and copper coloring

Matrix representations:

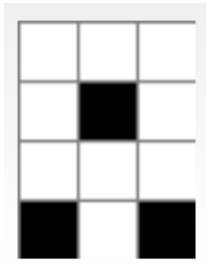
- A grayscale image is a 2-dimensional array of numbers. An 8-bit image has entries between _______ and ________.
- The value _______ represents a white color, and the value _______ represents a black color.
- Lower numbers translate to _______________ pixels, while higher numbers translate to _______________ pixels.
- For an image that has $m\ x\ n$ pixels (i.e., "picture elements"), we represent that image using a matrix of size _______________. The entries of the matrix indicate the pixel value of the corresponding part of the image.

Example: This table represents an image that has $4\ x\ 5$ pixels.

Example: This $4\ x\ 4$ image of the raccoon is represented by the matrix on the right below:

```
[[114 130 145 147]
 [ 83 104 123 130]
 [ 68  88 109 116]
 [ 78  94 109 116]]
```

Example: Write the matrix that represents the following image:

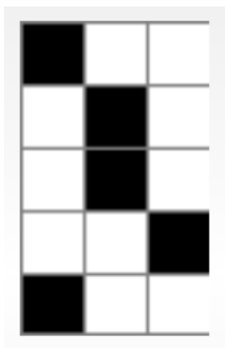


$$\begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix}$$

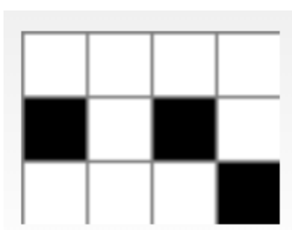
Exercises:

1. Write a matrix that represents each of the following images.

c.



d.



2. Draw a visual representation of the following matrices.

d. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

f. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$

Color Images:

Color images can be stored in a similar fashion to a grayscale image. Instead of one number (0-255) per pixel, one stores three numbers per pixel – these three numbers denote the "amount" of red, "amount" of green, and "amount" of blue in each pixel. These three numbers can be used to depict a wide range of colors.

Image transformations:

Once images are represented as matrices, we can describe many transformations of those images using basic matrix operations!

Example: Suppose we have two separate images that can be represented by the matrices $A = \begin{bmatrix} 255 & 0 \\ 0 & 255 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 255 & 0 \end{bmatrix}$.

4. Draw the images that correspond to A and B.

5. Find and draw $A + B$.

6. Compute $C = \frac{1}{5}A$. How would you describe the image represented by C in relation to the image represented by A?

Note: When mathematical operations on images result in entries that range outside of the integer values 0 through 255, image visualization tools treat them as 255 if the value exceeds 255, and as 0 if the value is below 0.

Exercises:

2. a. Describe an image whose matrix representation contains entries of all zeros.

b. Describe an image whose matrix representation contains entries of all 255.

3. Let A and B be matrices that represent two images of the same size. Describe in words what each of the following means.

   a. $A - B$

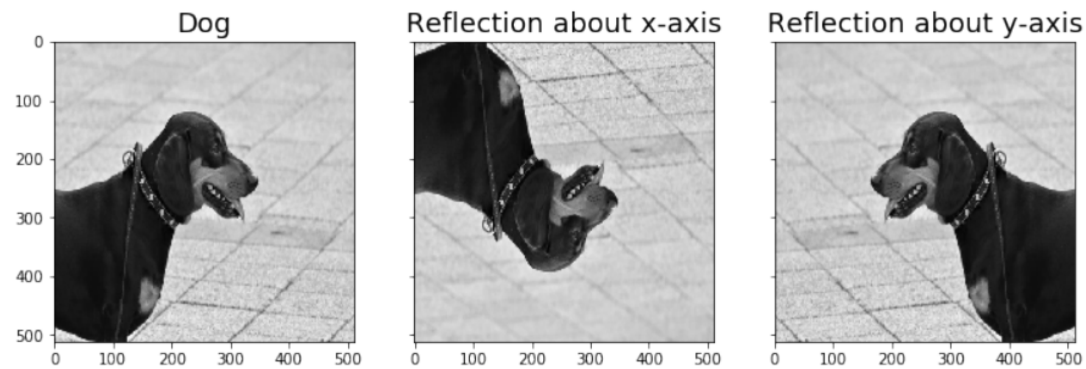
   b. $5A + B$

   d. $255 * 1 - A$, where "1" is the matrix of all 1's

Part II: Manipulating images

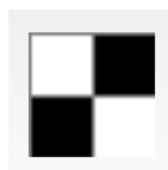
Reflecting/rotating images

Let us consider how we might reflect an image over the *x* or *y*-axis. Here's an example!

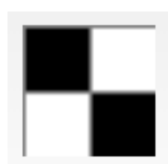


How would we represent these reflections using matrices?

From our earlier work, we know that the image represented by the matrix $X = \begin{bmatrix} 255 & 0 \\ 0 & 255 \end{bmatrix}$ would be:



If we were to reflect this image about the *x*-axis, we would obtain:



The matrix X' that represents this image is $X = \begin{bmatrix} 0 & 255 \\ 255 & 0 \end{bmatrix}$. We can generalize this result!

Exercises:

1. Which matrix represents the image that is the reflection of the matrix $A = \begin{bmatrix} 0 & a & b & 0 \\ c & d & 0 & 0 \\ e & 0 & f & g \\ 0 & 0 & 0 & h \end{bmatrix}$?

2. Write the matrix that corresponds to the reflection of each of the following about the
   c. $x -$ axis
   d. $y -$ axis

b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$ c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$

Reflecting images using matrix-matrix multiplication

Consider the matrix $X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ that represents an image[3].

Question: From our work above, we know that the matrix representing the image reflected about the x-axis is $\begin{bmatrix} & \end{bmatrix}$. What matrix would we need to multiply X by on the **left** in order to obtain the matrix of the image reflected about the $\boldsymbol{x}$**-axis**? Show that your answer gives you the correct result.

[3] (Note that these values each represent a shade of color between white and black, where 0 represents black, and 1 represents white. While we have used only values of 0 and 255 thus far, we will use these values in this example to make the reflections clear.)

Question: From our work above, we know that the matrix representing the image reflected about the y-axis is $\left[\qquad\qquad\right]$. What matrix would we need to multiply X by on the **right** in order to obtain the matrix of the image reflected about the $\boldsymbol{y}$**-axis**? Show that your answer gives you the correct result.

The matrix you used above is a form of a ____________________________ matrix (here referred to as a reverse identity matrix). This matrix is always ______________, and its size is determined by the size of the matrix it is multiplied to (i.e. in order for the matrix multiplication to be performed).

Exercises:

5. What size does the reverse identity matrix need to be to be used to reflect an image represented by a $5\ x\ 3$ matrix about the x-axis? Why?

6. What size does the reverse identity matrix need to be to be used to reflect an image represented by a $5\ x\ 3$ matrix about the y-axis? Why?

7. Use the reverse identity matrix of the appropriate size to reflect the images represented by the following matrices about the x-axis.

a. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$

8. Use the reverse identity matrix of the appropriate size to reflect the images represented by the following matrices about the y-axis.

a. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 0 & 255 \\ 0 & 255 & 255 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 255 & 255 \\ 0 & 255 & 255 \\ 255 & 0 & 0 \\ 255 & 255 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 255 & 0 & 0 \\ 0 & 255 & 0 \end{bmatrix}$

Reflecting images using matrix-vector multiplication

We can also manipulate images using matrix-vector multiplication. This method is a bit more powerful than matrix-matrix multiplication, because it allows us with more flexibility in ways you can manipulate those images.

Using this method, we first convert our matrices into vectors (i.e., single column or single row matrices).

- We define $vec(X)$ as the vector conversion of the matrix X. In general, for $X = \begin{bmatrix} \vdots & & \vdots \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}$, we have $vec(X) = \begin{bmatrix} \vdots \\ x_1 \\ \vdots \\ \cdots \\ \vdots \\ x_n \\ \vdots \end{bmatrix}$.
- We define $mat(X)$ as the matrix conversion of the vector Y. In general, for $Y = \begin{bmatrix} \vdots \\ x_1 \\ \vdots \\ \cdots \\ \vdots \\ x_n \\ \vdots \end{bmatrix}$, we have $mat(X) = \begin{bmatrix} \vdots & & \vdots \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}$.

Examples:

5. For $X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $vec(X) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

6. For $X = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $vec(X) = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$.

7. For $X = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 1 \end{bmatrix}$, $mat(X) = \begin{bmatrix} & \end{bmatrix}$.

8. For $X = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $mat(X) = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$.

Exercises:

3. Convert each of the following matrices into vectors.

   a. $A = \begin{bmatrix} 5 & 0 \\ 3 & 4 \end{bmatrix}$ $\qquad vec(A) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

   b. $B = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$ $\qquad vec(B) = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$.

c. $C = \begin{bmatrix} 8 & 8 & 8 \\ 7 & 7 & 7 \\ 8 & 8 & 8 \end{bmatrix}$ $\qquad vec(C) = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}.$

4. Convert each of the following vectors into matrices.

a. $X = \begin{bmatrix} 3 \\ 5 \\ 6 \\ 1 \end{bmatrix}$ $\qquad mat(X) = \begin{bmatrix} \quad \end{bmatrix}$

b. $Y = \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ $\qquad mat(Y) = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$

Reflection about the axes using matrix-vector multiplication

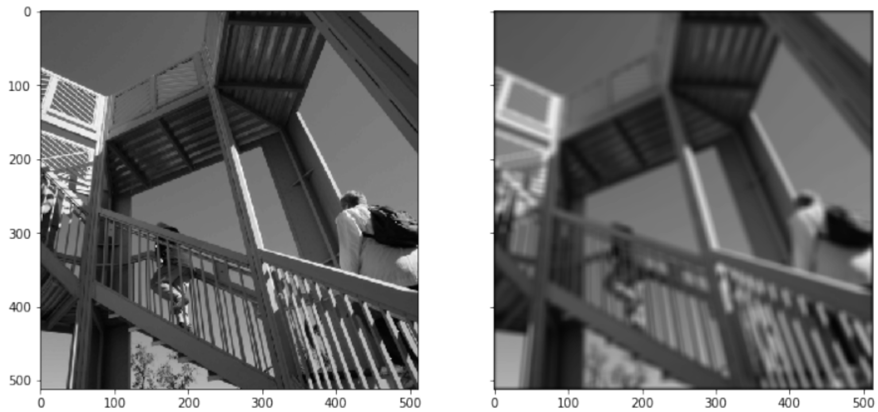
Consider the matrix $X = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. We know that $vec(X) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

Question: From our earlier work, we know that the reflection of the image with matrix X about the x-axis would be $Y = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$. So, $vec(Y) = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$. What matrix would we need to multiply $vec(X)$ by on the **left** in order to obtain $vec(Y)$? Show that your answer gives you the correct result.

Question: From our earlier work, we know that the reflection of the image with matrix X about the y-axis would be $Y = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$. So, $vec(Y) = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$. What matrix would we need to multiply $vec(X)$ by on the **right** in order to obtain $vec(Y)$? Show that your answer gives you the correct result.

Blurring images

We can also blur images to get an effect we're interested in, and we use linear algebra to do it! The image on the left below is the picture, and the image on the right is the blurred version.



One way to do this is to average each entry's "neighbors" (including the entry itself). The definition of these neighbors can be determined by the user themselves. For example, a neighbor could be any entry in the matrix representation that is directly left, right, above and below the original entry, along with the entry itself.

Example: Consider an image whose matrix representation is $X = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$. Suppose we define a "neighbor" as an entry that is directly above, below, next to or diagonal from that entry. For example, the entries that are neighbors of $x_{31} = 3$ in the matrix are $x_{21} = 2$, $x_{22} = 1$, $x_{32} = 1$, and $x_{31} = 3$. So the entry y_{31} in the new matrix would be $\frac{2+1+1+3}{4} = \frac{7}{4}$.

This gives us $Y = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ 7/4 & \quad & \quad \end{bmatrix}$.

Challenge Question: Find the $9 \; x \; 9$ matrix A such that $Y = A * vec(X)$. Show that your choice indeed results in Y.

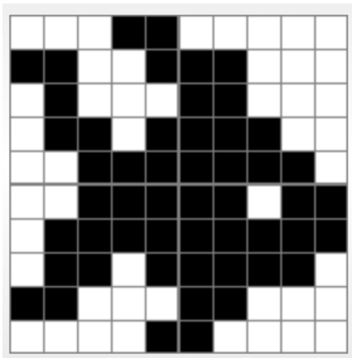
Lesson 5: Warm-Up – Student Version

- Student version begins on next page

Image Processing — Name: ____________________________

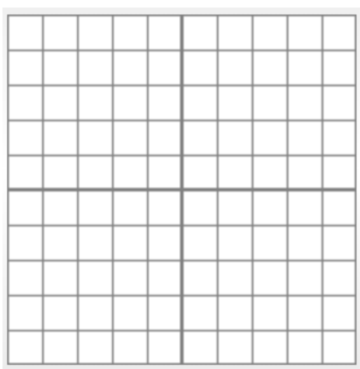
Warm-Up — Date: _________________ Period: ____

Given the image below, write the corresponding matrix:



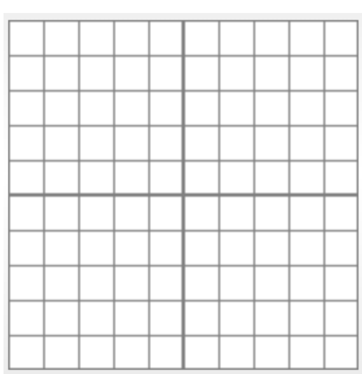
$A =$

Reflect the image over the x-axis, write the new matrix, and draw the new image:



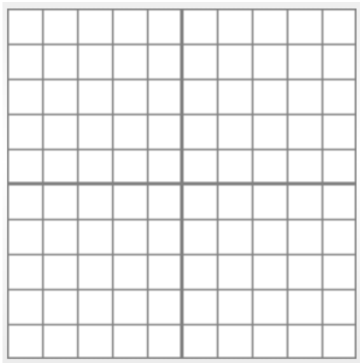
$A =$

Reflect the previous image over the y-axis, write the new matrix, and draw the new image:



$A =$

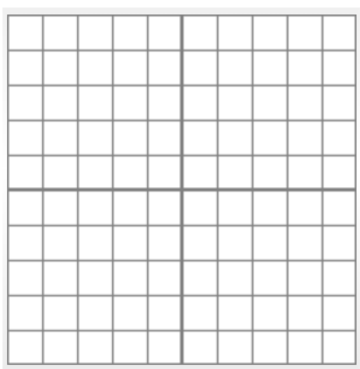
Reflect the previous image over the 45^{o} line, write the new matrix, and draw the new image:



$$A = \begin{bmatrix} \\ \end{bmatrix}$$

Change the brightness of the previous image by a scalar of 1/2, write the new matrix, and draw the new image:

*Note: you will use the color gray to draw the darkened or lightened image.



$$A = \begin{bmatrix} \\ \end{bmatrix}$$

- **Did you darken or lighten the image? Why?**

Blur the original image and write the new matrix (you do not need to draw the new image):

$$A = \begin{bmatrix} \\ \end{bmatrix}$$

Lesson 5: Instagram Project Outline

- ❖ The outline begins on the next page

Instagram Project

You are going to edit your given image to be Instagram ready! You will need to use a combination of reflections, blurs, and brightness changes to post this image and get LOTS of likes and comments.

Step 1:

- Come up with an Instagram handle (Note: all Instagram handles must be school appropriate and approved by the teacher before you can choose your image for step 2)

Step 2:

- Choose an image from your teacher to utilize throughout this project. First come first serve, so be sure to pick your image quick!

Step 3:

- As a group, decide what types of reflections, inversions, and brightness changes you would like to complete to post your picture. You **MUST** use one of each type of edit!

Step 4:

- Create matrices and a new image for each edit that you complete. You will need to turn this into your teacher at the end of the project.

Step 5:

- Once your edits are complete, you must come up with a perfect caption for this photo (Note: all captions must be school appropriate).
    - Remember, the better the caption, the more likely you are to receive more likes and comments!

Step 6:

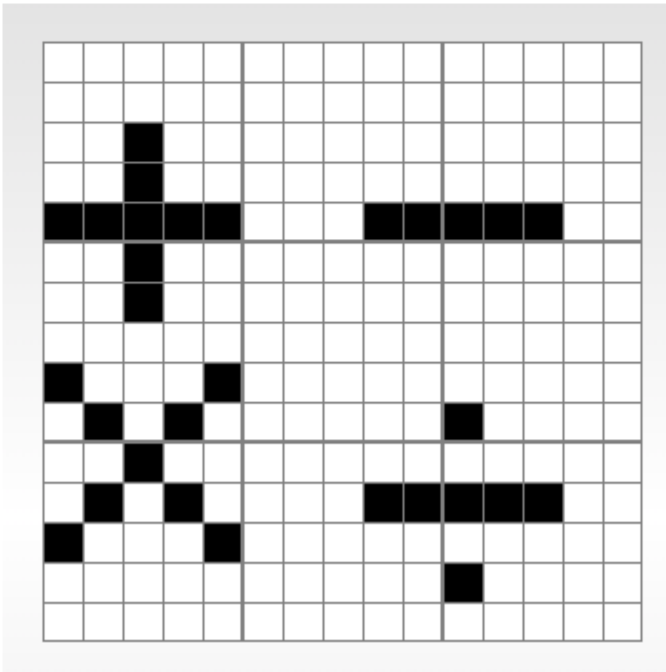
- Post your final image to the classroom Instagram feed on the wall and turn in all other materials and work to your teacher.
    - All other materials and work must be turned in in an organized manner with the names of each group member on them.

Grading Rubric:

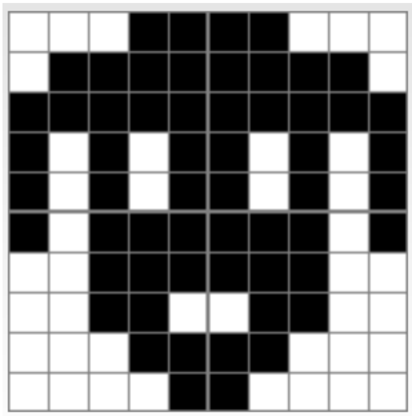
What the teacher is looking for:	**Potential Points Earned**	**Actual Points Earned**
1. Did the student follow the directions?	1	
2. Does the student show understanding of how to reflect the image?	4	
3. Does the student show understanding of how to invert an image?	4	
4. Does the student show understanding of how to adjust the brightness of an image?	4	
5. Did the student complete their math correctly?	2	
Total Points:	15	

Appendix B: Lesson Materials – Templates/Examples

Lesson 4: Math Symbols Image



Lesson 5: Instagram Project Stock Images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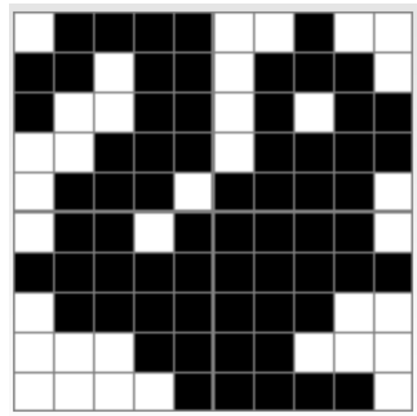
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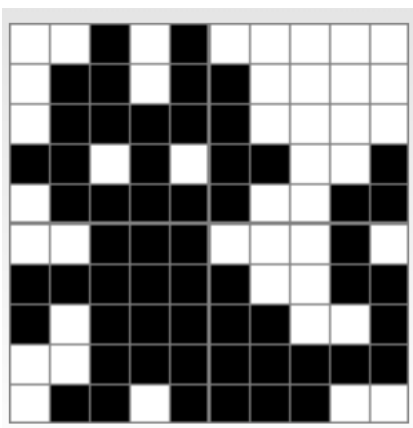
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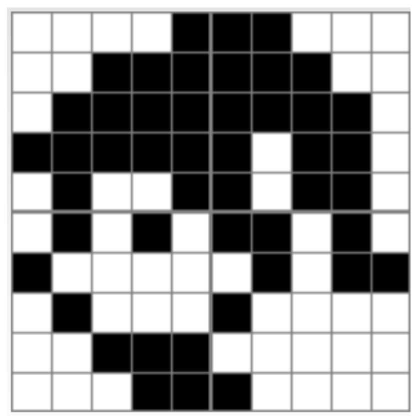
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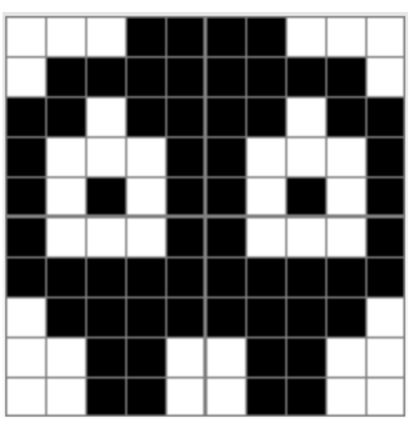
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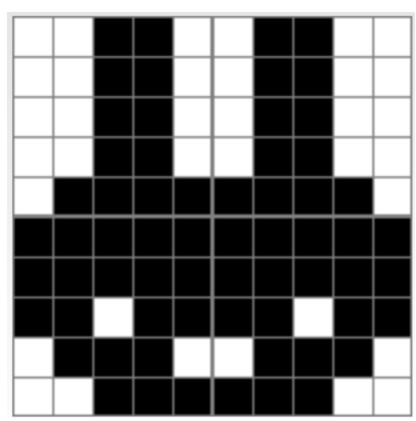
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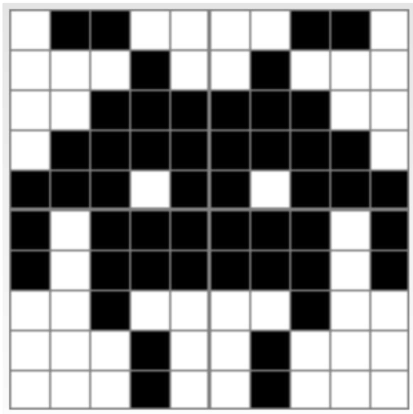
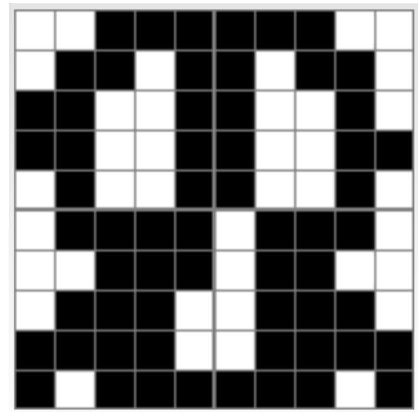
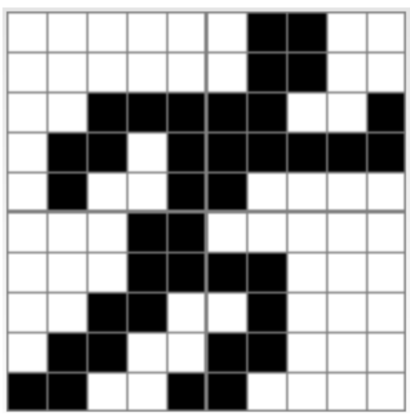
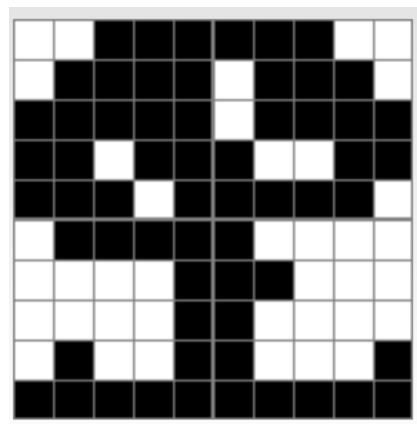
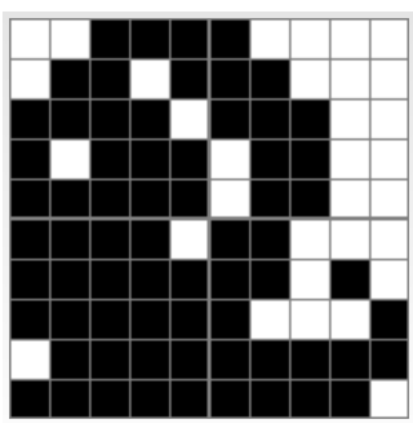
 **

 **

 **

 ***

 ***

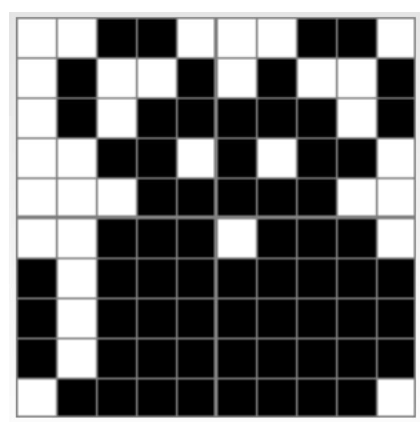
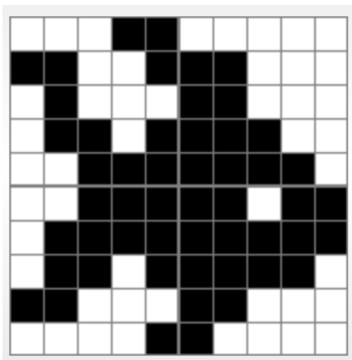
 ***

*Image Source: https://www.woojr.com/printable-picross-grid-puzzles/
**Image Source: https://play.google.com/store/apps/details?id=org.popapp.jc&hl=en_US
***Image Source: https://nonogramskatana.wordpress.com/tag/10x10/

Lesson 5: Instagram Project Exemplar

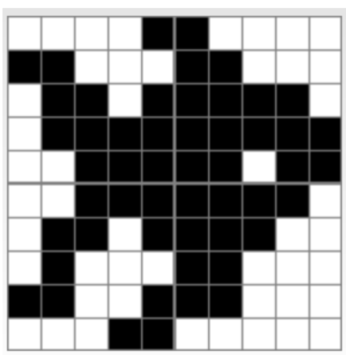
Instagram Handle: @mathpeeps314

Original Image:



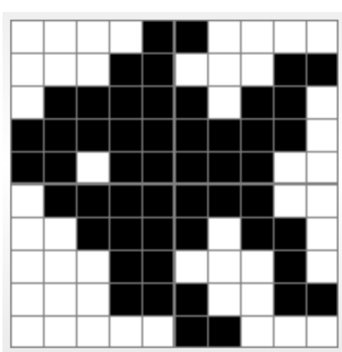
$$\boldsymbol{A} = \begin{bmatrix} \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} \end{bmatrix}$$

Reflect the image over the x-axis:



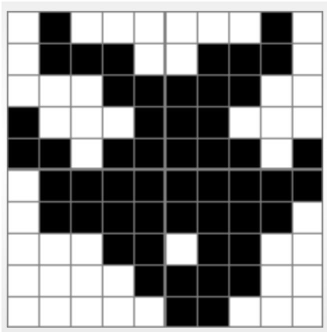
$$\boldsymbol{A} = \begin{bmatrix} \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} \\ \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} \end{bmatrix}$$

Reflect the previous image over the y-axis:



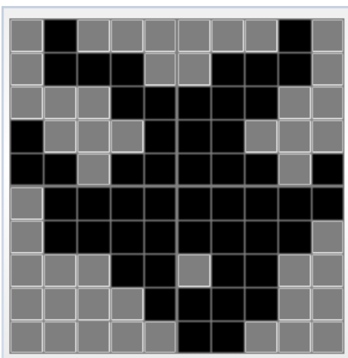
$$\boldsymbol{A} = \begin{bmatrix} \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{255} \\ \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} \\ \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{255} & \mathbf{0} & \mathbf{0} & \mathbf{255} & \mathbf{255} & \mathbf{255} \end{bmatrix}$$

Reflect the previous image over the 45^{o} line:



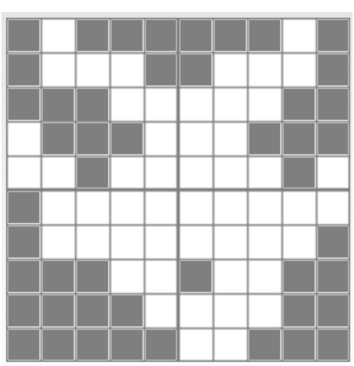
$$A=\begin{bmatrix}255&0&255&255&255&255&255&255&0&255\\255&0&0&0&255&255&0&0&0&255\\255&255&255&0&0&0&0&0&255&255\\0&255&255&255&0&0&0&255&255&255\\0&0&255&0&0&0&0&0&255&0\\255&0&0&0&0&0&0&0&0&0\\255&0&0&0&0&0&0&0&0&255\\255&255&255&0&0&255&0&0&255&255\\255&255&255&255&0&0&0&0&255&255\\255&255&255&255&255&0&0&255&255&255\end{bmatrix}$$

Change the brightness of the previous image by a scalar of 1/2, write the new matrix, and draw the new image: *Note: you will use the color gray to draw the darkened or lightened image.



$$A=\begin{bmatrix}\frac{255}{2}&0&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&0&\frac{255}{2}\\\frac{255}{2}&0&0&0&\frac{255}{2}&\frac{255}{2}&0&0&0&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&0&0&0&0&0&\frac{255}{2}&\frac{255}{2}\\0&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&0&0&0&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}\\0&0&\frac{255}{2}&0&0&0&0&0&\frac{255}{2}&0\\\frac{255}{2}&0&0&0&0&0&0&0&0&0\\\frac{255}{2}&0&0&0&0&0&0&0&0&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&0&0&\frac{255}{2}&0&0&\frac{255}{2}&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&0&0&0&0&\frac{255}{2}&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&0&0&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}\end{bmatrix}$$

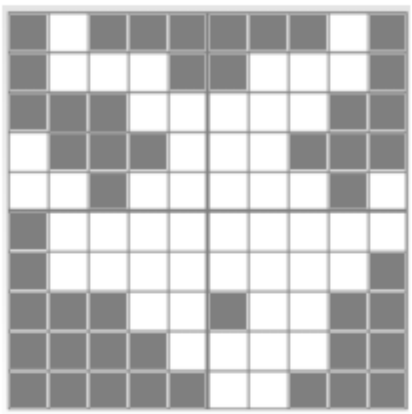
Image Inversion – $255\cdot\begin{bmatrix}1&\cdots&1\\\vdots&\ddots&\vdots\\1&\cdots&1\end{bmatrix}-A$**:**



$$A=\begin{bmatrix}\frac{255}{2}&255&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&255&\frac{255}{2}\\\frac{255}{2}&255&255&255&\frac{255}{2}&\frac{255}{2}&255&255&255&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&255&255&255&255&255&\frac{255}{2}&\frac{255}{2}\\255&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&255&255&255&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}\\255&255&\frac{255}{2}&255&255&255&255&255&\frac{255}{2}&255\\\frac{255}{2}&255&255&255&255&255&255&255&255&255\\\frac{255}{2}&255&255&255&255&255&255&255&255&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&255&255&\frac{255}{2}&255&255&\frac{255}{2}&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&255&255&255&255&\frac{255}{2}&\frac{255}{2}\\\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}&255&255&\frac{255}{2}&\frac{255}{2}&\frac{255}{2}\end{bmatrix}$$

Final Product:

mathpeeps314



Mathpeeps314 Here fishy fishy fishy